Probability, pendulums, and pedagogy

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Deterministic pendula exhibit a spectrum of behavior ranging from periodic to chaotic and provide an opportunity for an introductory discussion on the application of probability techniques to a deterministic system. Analytic and simulation techniques are used to determine probability distributions for a range of dynamical possibilities. In particular, we obtain probability distributions of the pendulum's angular displacement and distributions of first return times for regular and chaotic motion. For chaotic motion, the latter distribution is modeled by a simple two-state Bernoulli process. Further considerations suggest that not all distributions are probability distributions.

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I. INTRODUCTION

Students of introductory and intermediate physics are mostly exposed to deterministic physical systems and only occasionally to probabilistic systems. Probability models may therefore seem nonintuitive, and students might feel lost when grappling with probabilistic concepts. In this paper, we present a discussion of probability in the context of a simple physical system, the pendulum.

The classical deterministic pendulum is an apparently simple system, but can exhibit rich physical behavior.[1–3] Remarkably, probability techniques can characterize some of this behavior despite the deterministic nature of the system. The application of probability techniques to a deterministic system might seem strange and even inappropriate. Yet such a seeming contradiction can help to sharpen student understanding of the concepts of determinism, randomness, and probability. In this paper, we present simulations of the deterministic pendulum and various probability models for increasingly complex motion of the pendulum. These models are then discussed with reference to two common definitions of probability.[4]

The motion of the free, undamped, classical deterministic pendulum is periodic. If friction and a periodic external force are added, the pendulum can exhibit both regular oscillations and chaotic behavior. In the latter case, the motion is characterized by extreme sensitivity to the initial state and any small variation in this state will quickly lead to a very different final state.[5,6] Thus, chaotic systems can have the appearance of random behavior. More precisely, there is a hierarchy of randomness in chaotic systems,[7] ranging from ergodic systems, where time averages equal state space averages,[8] to the most random category, Bernoulli systems.[9] The chaotic pendulum exhibits a degree of randomness and occupies a midposition on this scale. Therefore, a probability model appropriate to an idealized random system can describe some aspects of a pendulum's chaotic behavior.

We start from the simplest periodic motion of the linearized pendulum and obtain, analytically and by simulation, probability distributions of the angular displacement for increasingly complex configurations of the pendulum. We also obtain the distribution of times for which the pendulum returns to an original state, the *first return time*.[10] Both distributions distinguish between systems that are manifestly deterministic (regular) and those that, although still deterministic, are apparently random (chaotic.) A simple, two state Bernoulli system is used to model the return time distribution in the chaotic case. Thus, we demonstrate both types of distributions as the pendulum motion becomes more complex and show the consequences of increasingly more randomlike behavior. Chaotic data thus provides a basis for the discussion of the use of probability in a deterministic, but apparently random system.

The paper is structured as follows. In Sec. II, we consider the small angle approximation of a simple pendulum and derive the well-known probability distribution for the angular displacement. This case is the only one for which the angular distribution can be calculated analytically. The distribution of first return times (to the down position) is even simpler: A spike located at a time equal to one-half of the period of the pendulum. The sharpness of this distribution is clear evidence of regular deterministic behavior. In Sec. III, we consider the nonlinear sinusoidal dependence of the gravitational restoring force and obtain the appropriate probability distributions using computer simulation techniques.[11] The power spectra show that, although still periodic, the motion contains harmonics of the basic frequency. Yet there is little qualitative difference between this distribution and the distribution for the linearized case, even for large amplitudes.

The pendulum is then made somewhat more complex by adding damping and forcing. Forcing adds a third degree of freedom to the system and the possibility now exists of period doublings and chaos. In Sec. IV, we explore angular and return time distributions of these motions with a particular emphasis on period doubling as a route to chaos. The probability aspects of the fully chaotic pendulum are also discussed. In Sec. V, we discuss two meanings of probability and the results of the previous sections in this context to determine the appropriateness of a probabilistic description of a deterministic physical system. The conclusion is in Sec. VI.

II. LINEARIZED PENDULUM

The equation of motion for the simple pendulum may be written as

$$\frac{d^2\theta}{dt^2} + \omega_0^2 \sin\theta = 0, \quad \theta(0) = \theta_0 \quad \text{and} \quad \dot{\theta}(0) = 0. \tag{1}$$

For small angles, $\sin\theta \approx \theta$, and the equation of motion becomes that of a linear, harmonic oscillator,

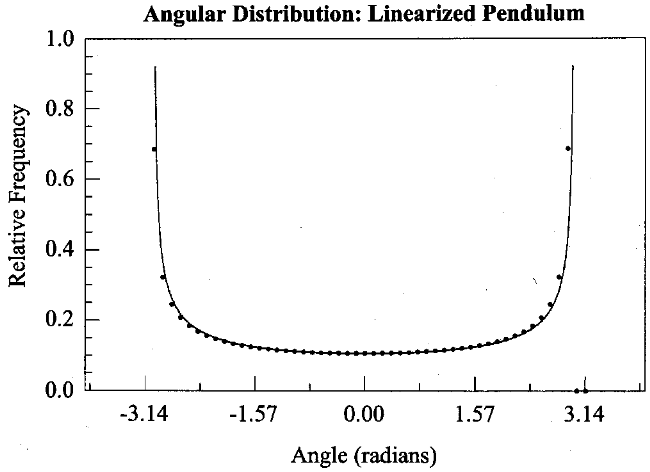

Fig. 1. Angular distribution of the linearized pendulum. The solid line is the analytic expression and the dots are obtained from the Monte Carlo sampling.

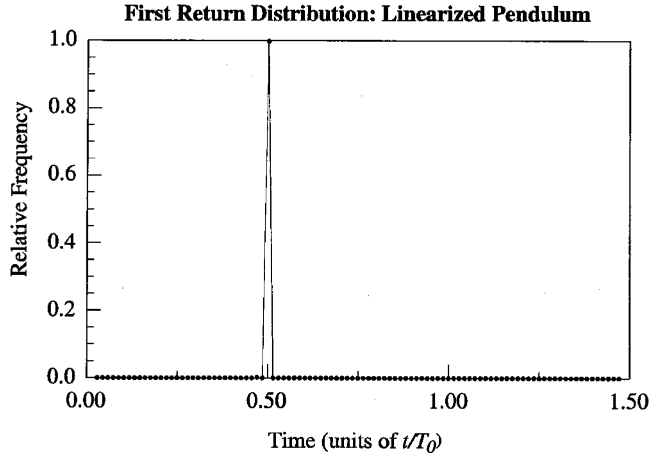

Fig. 2. The distribution of first return times to the angular interval of $\Delta\theta = 0.01$ radians centered at $\theta=0$, for the linearized pendulum. Because the return angle is $\theta=0$, there is only one peak observed at a time equal to one-half of the period. T_0 is the period of the pendulum.

$$\frac{d^2\theta}{dt^2} + \omega_0^2\theta = 0, \tag{2}$$

with the solution $\theta(t) = \theta_0 \cos \omega_0 t$. This system is manifestly deterministic, and it is easy to obtain an expression for the time the pendulum spends in a small angle between θ and $\theta + d\theta$. The amount of time dt spent in $d\theta$ is given by

$$dt = \left|\left(\frac{dt}{d\theta}\right)\right| d\theta = \left|\left(\frac{1}{d\theta/dt}\right)\right| d\theta = \frac{d\theta}{\omega_0\sqrt{\theta_0^2 - \theta^2}}. \tag{3}$$

We provisionally interpret this expression as a probability density function by assuming that the amount of time the pendulum spends in a certain angular interval is proportional to the probability of the system being in that interval—a sort of ergodic hypothesis. For a suitable normalization, $\int_{-\pi}^{\pi} P(\theta) d\theta = 1$, the density function becomes

$$P(\theta) = \frac{1}{\pi\sqrt{\theta_0^2 - \theta^2}}, \tag{4}$$

as shown by the solid line in Fig. 1.

One way to conceptualize the physics of this function is as follows. Suppose that a pendulum, moving from almost the upward position on one side to a symmetrical upward position on the opposite side, is exposed to a strobe light of uniform frequency. The density of images of the pendulum's bob will be variable: Higher in the upward position and lower near the downward position. Similarly Fig. 1 shows that the pendulum is more likely to be in an upward position where its motion is slow compared to a downward position where its motion is faster. Another interpretation would be that of a casual sporadic observer. If the observer looks at the pendulum at random times, then the graph of Fig. 1 gives the probability associated with each angular interval. But in this case, randomness, and therefore probability, are injected by the actions of the observer, and not by the deterministic physical system. The distribution itself is not intrinsically a probability distribution because the time that the pendulum arrives in any small interval is predictable.

The geometric shape of Fig. 1 can be interpreted in a way that is more in keeping with a probability by using data from a Monte Carlo sampling of the angular displacement as indicated by the dots of Fig. 1. We generate 100,000 values of the time at random using a linear random number generator that provides a uniform distribution of random times. From Eq. (2) this random set of times leads to random values of the angle θ. The number of "hits" in a given angular interval is sorted into the bins corresponding to each interval. The relative numbers of hits in each of the 50 bins are plotted and superposed on the solid line in Fig. 1, thus showing that the analytic result and the Monte Carlo sampling give the same result. It is important to note that because the pendulum is periodic, the 100,000 samples need not be random, but could also be an ordered temporal sequence of points. The result would be the same because the system is deterministically periodic. Therefore, although Eq. (4) appears to be a probability density function, it only indicates the amount of time the pendulum spends in a particular small angular interval and not the actual probability that, at any given time, the pendulum is in the given interval.

Similarly, the first return time of the pendulum, for example near $\theta=0$, occurs at regular time intervals and therefore the return time distribution is a delta function at a time equal to one-half of the period of the motion, as shown in Fig. 2. (If the interval of return were other than at $\theta=0$ there would be two distinct peaks.) This sharp peak is another clear indication that the pendulum's motion is regular and deterministic, and that the use of a distribution in this case is a formal device rather than a representation of reality.

III. THE NONLINEAR PENDULUM

We reintroduce the nonlinearity into the pendulum's equation of motion:

$$\frac{d^2\theta}{dt^2} + \omega_0^2 \sin\theta = 0, \tag{5}$$

whose solution $\theta(t)$ is periodic but no longer a single sinusoid. (The period of motion grows with increasing amplitude.) One way to approach the solution to Eq. (5) is by an iterative process, whereby an initial trial solution is assumed and then improved through substitution into the equation of motion. We take $\theta = A \cos \omega_0 t$ as the initial solution, substi-

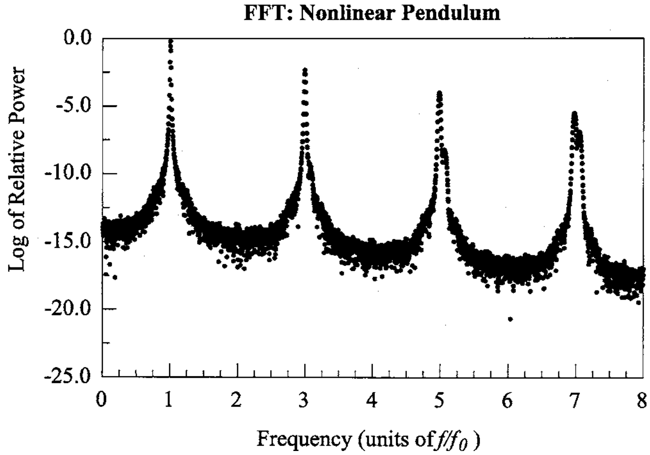

Fig. 3. Power spectrum for the time series of the angular displacement of the nonlinear pendulum [from a numerical solution to Eq. (5)]. Note the presence of the odd harmonics of the fundamental frequency ($f_0 = \omega_0 / 2\pi$), the largest peak.

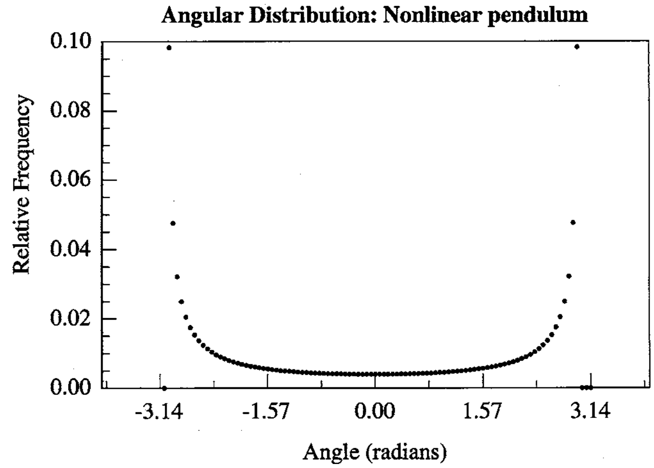

Fig. 4. Angular distribution of the nonlinear pendulum. The amplitude is 3 radians. The bin counting technique is used. Note the slightly different shape from Fig. 1.

tute it into Eq. (5), and then expand the sin θ term in a power series:

$$\frac{d^2\theta}{dt^2} = -\,\omega_0^2 \sin(A \cos \omega_0 t)$$
$$= -\,\omega_0^2 \left[A \cos \omega_0 t - \frac{(A \cos \omega_0 t)^3}{3!} + \frac{(A \cos \omega_0 t)^5}{5!} \cdots \right]. \tag{6}$$

Each of the powers of the trigonometric functions can be shown to contain a harmonic that corresponds to that particular power. Therefore, there are sinusoidal terms in the time series with frequencies that are odd harmonics of the fundamental frequency. For the next iteration, we might use $\theta = B \cos \omega_0 t + C \cos 3\omega_0 t$.

Figure 3 illustrates the reality of multiple harmonics in the motion of the nonlinear pendulum. This figure shows a power spectrum of the numerical solution to Eq. (5) with several discrete contributions. The largest contribution is at the fundamental frequency with successively smaller contributions at the higher odd harmonics. Unlike the linearized pendulum, the analytic inversion of Eq. (5) to find a probability density, $dt/d\theta$, is not possible. Instead, Eq. (5) can be solved numerically and each solution point is assigned to a bin covering the region $-\theta_0 < \theta < \theta_0$. A typical distribution of points is shown in Fig. 4. Given the large amplitude (3 radians) used in the generation of this distribution, it is perhaps surprising that the shape of the distribution of Fig. 4 is little different from the distribution of the linearized pendulum shown in Fig. 1. The return time distribution (to $\theta = 0$) consists, as in the linearized case, of a single peak, thereby affirming the regular deterministic motion of the pendulum.

IV. THE DAMPED DRIVEN PENDULUM

A. Regular motion

We now consider the damped, driven pendulum whose dissipative losses are replaced by the injection of energy by periodic forcing. For simplicity we assume that the forcing has the form of a single sinusoid. Furthermore, by choosing a time scale in units of $2\pi/\omega_0$ and a torque measured in units of the critical torque mgl (required to maintain the pendulum in a stationary horizontal position), the equation of motion is written as

$$\frac{d^2\theta}{dt^2} + \frac{1}{q}\frac{d\theta}{dt} + \sin \theta = F \cos \omega_D t, \tag{7}$$

where q is the reciprocal of the damping factor, ω_D is the drive frequency, and F is the amplitude of the forcing term. (In these units, the resonant angular frequency of the corresponding low amplitude pendulum is unity.) Sets of values of the parameters (q, ω_D, F) are points in a three-dimensional parameter space. A particular point in parameter space determines the motion of the pendulum. Its behavior may range from regular motion, characterized by the single frequency of the external force, to complex chaotic motion with a continuum of frequencies.

These motions may be represented in a bifurcation diagram as shown in Fig. 5 in which the amplitude F, indicated along the horizontal axis, is varied while the other param-

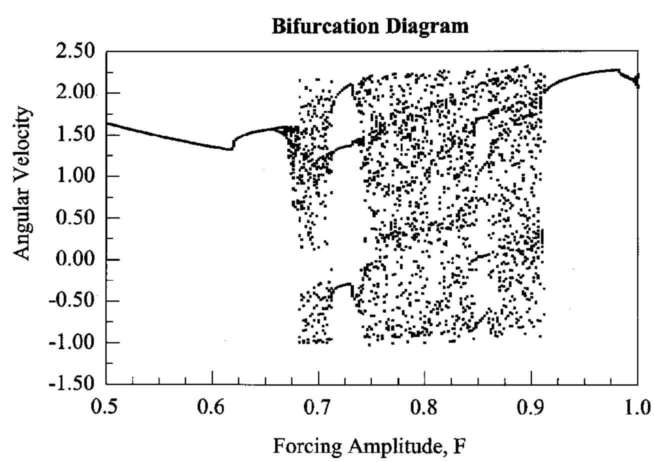

Fig. 5. A bifurcation diagram showing the regular and chaotic behavior of the pendulum. The angular velocity is sampled once every forcing cycle. Period doubling occurs at forcing amplitudes where two points occur in the region $0.658 < F < 0.666$. Chaos is evident when many points occur. The other pendulum parameters are $q = 4, \omega_D = 2/3$.

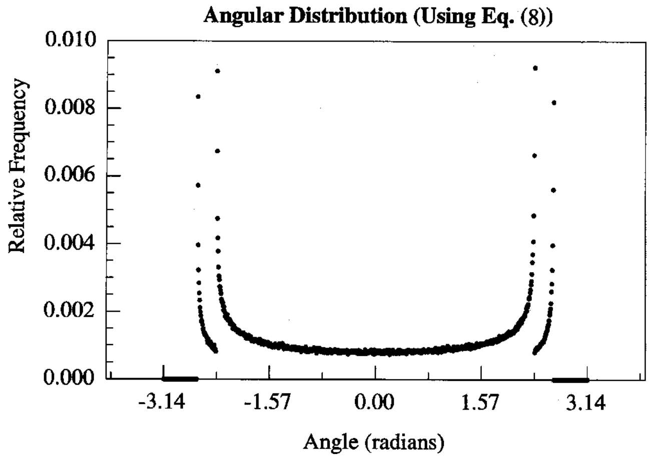

Fig. 6. The fundamental (amplitude=2.5 rad) and one subharmonic (amplitude=0.2 rad) generate the angular distribution given by Eq. (8).

eters are kept constant. (Other bifurcation diagrams may be generated using the other parameters.) The vertical axis shows repeated sampling of the pendulum's angular velocity at a particular phase point once during the forcing cycle. For some values of F, the graph only shows a single point, indicating that the motion is periodic with a period corresponding to that of the external force. For some other values of F, two points appear, indicating that the pendulum requires two forcing periods to make a complete periodic motion.[12] This transition to periodic motion at one-half of the forcing frequency is called period doubling and occurs for $0.658 < F < 0.666$. For larger F, there is a larger but still finite number of points, indicating more complex motion, with period corresponding to the number of points. Finally, for sufficiently large F, the pendulum's motion is chaotic, moving endlessly from one unstable periodic orbit to another.

We now consider period two behavior for which the pendulum takes two forcing periods to complete one cycle. It may be approximately described by periodic motion with two terms: One at the forcing frequency and one at half the forcing frequency:

$$\theta(t) = A_1 \cos \omega_D t + A_{1/2} \cos(\omega_D/2)t. \tag{8}$$

Monte Carlo sampling is used to develop the angular distribution $P(\theta)$ shown in Fig. 6, with amplitudes stated in the Fig. 6 caption. This distribution shows two peaks arranged symmetrically on either side. The power spectrum of Fig. 7(a) shows that more than the two harmonic terms suggested by Eq. (8) are required for an accurate distribution. To incorporate these further harmonics, the sampling is now applied to the numerical solution of the equation of motion, Eq. (7), with the results shown in Fig. 7(b). Like Fig. 6, the primary feature is the pair of peaks, but unlike the two-term sampling, the distribution is slightly asymmetric, which indicates that the pendulum's motion is not symmetric about its vertical axis and is determined in part by the initial phase coordinates of the pendulum's motion. That is, depending on the initial angular position and velocity, the motion will be in one of two possible period-2 orbits.[12]

The distribution of first return times for the period 2 pendulum is somewhat more complicated than that of the linearized pendulum which had only one spike. The time series is slightly offset from $\theta=0$ (the asymmetry mentioned above),

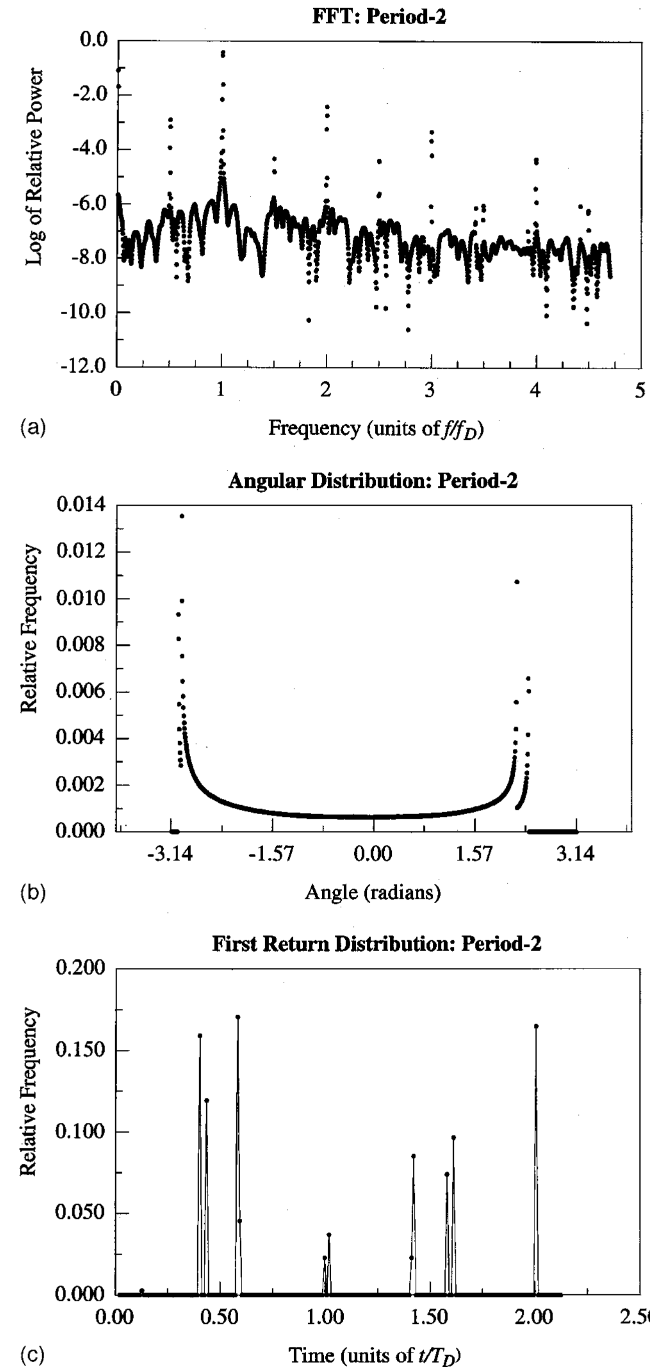

Fig. 7. Period doubling. $q=4, \omega_D=2/3, F=0.664$ (a) Semilog power spectrum of a period doubled time series of the angle. The doubling is shown by the peak at about $f=0.5 f_D$ which is half of the fundamental forcing frequency, that is indicated by the largest peak at about $f=f_D$. Other peaks occur at half-integral values of the forcing frequency. (b) Angular distribution for period doubling. Note the double singularities at the edges of the distribution. (c) The distribution of first return times. Note that the range of the data is about double the forcing period, T_D.

the range over which the distribution occurs is double the period, and the combination of both effects yields several peaks within a time equal to $t_{\max}=4\pi/\omega_D$, as illustrated in Fig. 7(c).

We expect that if further period doubling or period tripling occurs, the distribution will have further complexity. As an example consider the bifurcation diagram shown in Fig. 8(a)

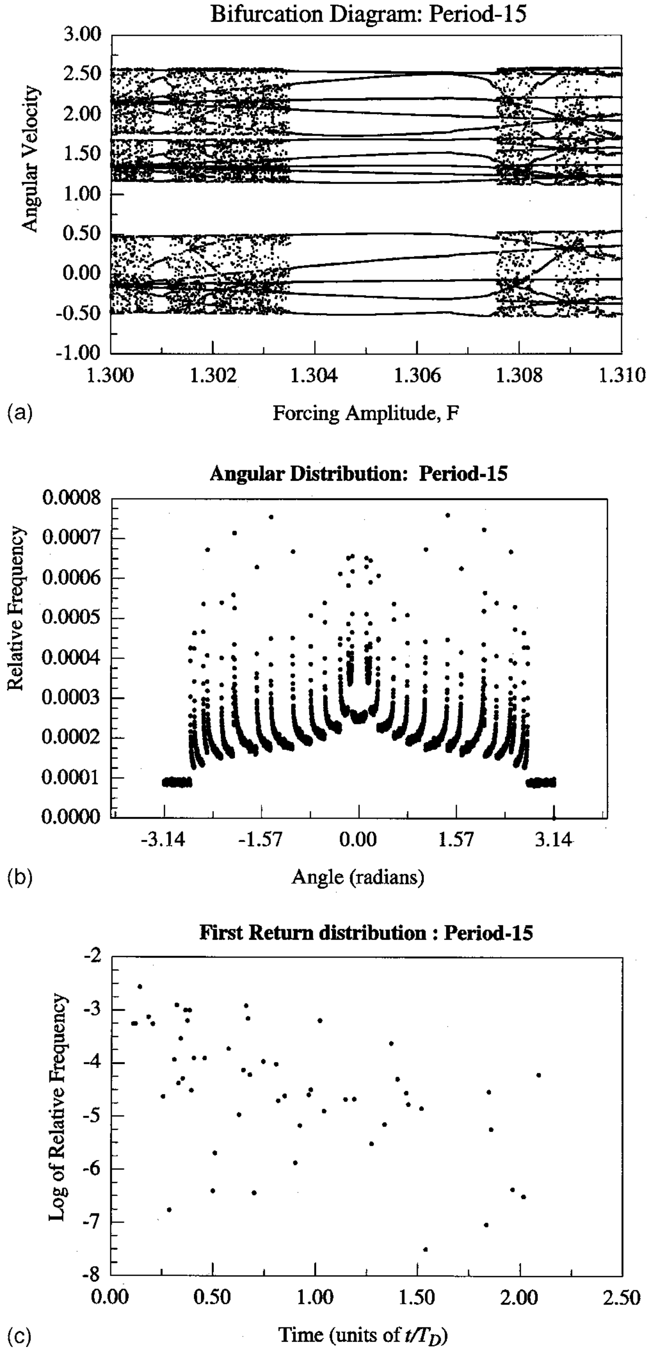

Fig. 8. Period-15 motion. $q=4, \omega_D=2/3$. (a) A bifurcation diagram near the regime of period-15 motion. $1.299 < F < 1.311$. (b) Angular distribution in which the 15 singularities on either side of $\theta=0$ indicate the presence of period-15 motion. $F=1.305$. (c) Semilog plot of the first return distribution, $F=1.305$. The time is in units of the forcing period, T_D.

which illustrates a regime where the motion is regular yet quite complex. The periodic window in the region of $F=1.305$ has a period 15 times that of the forcing period. This complexity is represented in the distribution function for the angular displacement as shown in Fig. 8(b). The distribution now exhibits many singularities and a careful count shows 15 singularities on either side of $\theta=0$.

The complexity of the probability distribution for the angular displacement suggests that despite the regularity of the motion, the pendulum returns to a fixed interval such as $0.0 \leq \theta \leq 0.01$ at a variety of different times. The return time distribution shown in Fig. 8(c) demonstrates this behavior.

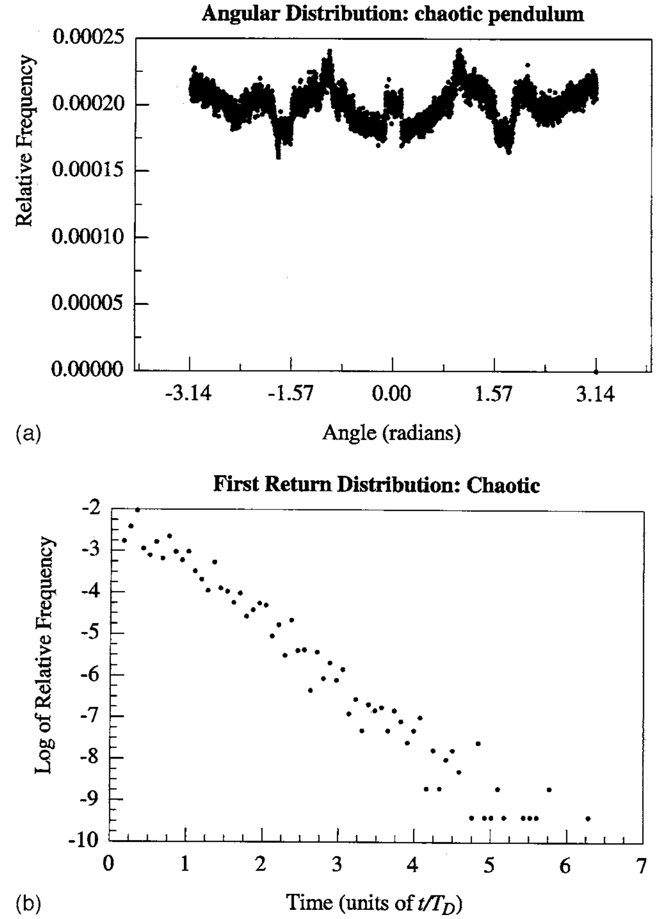

Fig. 9. Chaotic motion, $q=4, \omega_D=2/3, F=4$. (a) Angular distribution. The many peaks of the period-n regular motion have now coalesced into a bumpy, but approximately rectangular distribution. The distribution is apparently symmetric around the $\theta=0$. (b) First return distribution, to the interval $|\theta-\theta_1|=|0-0.01|$ rad. The exponential decay model approximates the data.

B. Chaotic motion

The transition to chaotic motion is often due to an increasing number of period doublings.[13] The motion becomes so complex that a transition occurs to an infinite number of periodic, but unstable, orbits. The system now becomes sensitive to initial conditions and is chaotic. The singularities of the distribution shown in Fig. 8(b) grow in number such that they merge, and the distribution for the chaotic pendulum becomes bumpy with a slight rise toward $\theta=0$, as shown in Fig. 9(a). The distribution is approximately rectangular, and the pendulum spends roughly equal times in each angular interval. Yet there are bumps. What do they signify? The motion of the pendulum may be represented on an attractor in the phase space whose coordinates are $(\theta, \dot{\theta})$. For regular motion the shape of the attractor is that of a periodic limit cycle. For chaotic motion, the shape of the pendulum attractor is of a fairly dense attractor, yet the density of points (from a discrete time simulation) is variable and therefore the rectangular distribution does have some bumps. The fact that it is uniformly dense is confirmed by the fact that the distribution is approximately rectangular and the bumpiness does not change significantly even when the number of counting bins is significantly increased. Thus the pendulum spends roughly equal times in all segments along the θ axis of the phase space.

Similar considerations apply to the return time distribution. As noted, chaotic motion is characterized by the existence of an infinite number of unstable periodic orbits. That is, the pendulum spends a brief time in one periodic orbit and then jumps to another periodic orbit for another brief time. The time spent on a given unstable orbit is variable. The motion may seem regular for a finite time before a series of short term visits to a collection of unstable orbits. In other words, some unstable orbits are less unstable than others. Thus, for some choices of the return location θ, the distribution of the first return times may have a few spikes, but rather than being sharp as for regular motion, the spikes decay exponentially. For other choices of θ, the motion is more random with no obvious spikes and the distribution of return times is approximately exponential. This exponential decay behavior can be modeled by a simple random process, as follows.

Consider a set of Bernoulli trials for which two states are possible. One state occurs when the pendulum lands in a particular (small) angular interval $\Delta\theta$ about θ and the probability of this event is μ. If the interval is small, then μ is small. The other state occurs when the pendulum lands in any other angular interval with probability $1-\mu$. For the return time distribution to the particular interval we require the probability that the pendulum angle has a value close to θ given that t time units has occurred since the last time the pendulum was near this value; that is, $P_{\Delta\theta}(t:\mu)=\mu(1-\mu)\times(1-\mu)\ldots(1-\mu)$ for $(t-1)$ time units. This expression may be rewritten as

$$P_{\Delta\theta}(t;\mu) = \mu(1-\mu)^{(t-1)} = \frac{\mu}{1-\mu}e^{t\,\ln(1-\mu)} \sim \mu e^{-\mu t} = \frac{e^{-t/\tau}}{\tau}, \tag{9}$$

as $\mu = 1/\tau$ becomes very small. Therefore, the distribution of first return times is exponential and is characterized by an average decay time τ because

$$\langle t\rangle = \int_0^\infty tP_r(t)dt = \tau. \tag{10}$$

Figure 9(b) shows the first-return time distribution of the chaotic pendulum from a numerical simulation. In this case, the semilogarithmic plot indicates that single exponential behavior is dominant as suggested by the simple model. The first return statistics give a strong indication that despite its deterministic equation of motion, the pendulum acts stochastically.[14]

V. DISCUSSION

We have studied a range of behavior from the most regular and simple, as for the linearized pendulum, to the complex behavior of the nonlinear chaotic pendulum. Our purpose is to use a concrete physical system to encourage discussion of the meaning and use of probability distributions for deterministic systems. The subject of probability as applied to chaotic systems is complex. A hierarchy of randomness exists in chaotic dynamics. The distinctions are subtle and require sophisticated mathematical logic, well beyond the scope of this paper.[15] However, a few comments can be made that may help students in thinking about probability and determinism.

The economist John Maynard Keynes described probability as a primitive term and wrote that "A definition of probability is not possible… We cannot analyze the probability-relation in terms of simpler ideas."[16] Despite this view Roy Weatherford has noted the existence of a classification scheme that includes eleven different categories of meanings of probability.[17] For this introductory discussion, we follow the more usual and simpler schema of two broad categories of meaning for probability.[18]

The first definition of probability, often attributed to Pierre Simon LaPlace and Daniel Bernoulli, is from the classical theory of probability, and is sometimes called the classical definition. The observer makes an *a priori* estimate or expectation of a certain favorable event. A given event may occur in several ways, each way being an outcome. If all outcomes are equally probable, the probability of a given event is the ratio of the number of ways (outcomes) the favorable event can occur to the number of ways that all events (and thus all outcomes) can occur. That is

$$P(favorable\ event) = \frac{Number\ of\ possible\ favorable\ outcomes}{Number\ of\ possible\ outcomes}. \tag{11}$$

The favorable event might be the system landing in a certain interval in some sort of system state space. The outcomes might be "cells" in the state space.

Another definition of probability, sometimes called the frequency definition, utilizes the frequency of occurrence of the favorable event in an experiment, and if the favorable event can occur in several ways, then probability is defined as

$$P(favorable\ event) = \frac{Frequency\ of\ favorable\ outcomes}{Frequency\ of\ all\ outcomes}. \tag{12}$$

We remind ourselves that the pendulum is a deterministic system. This fact has consequences for the classical definition of probability. In this definition, is assumed that at any time all outcomes are equally possible. For the deterministic regular or periodic pendulum, the probability that the pendulum's angular displacement is within a certain range about θ at any given time is either 1 or 0. That is, the pendulum's angular displacement is either in a certain interval or it is not, and there is no fractional probability because the pendulum's angular displacement is well known at any time. There is no information entropy.[19] Thus, the use of classical probability for the regular periodic pendulum seems inappropriate if probability is defined as in Eq. (11).

On the other hand, the frequency definition of probability is applicable to the periodic pendulum. Indeed, there is a certain frequency of occurrences for a given angular interval and it seems appropriate that we can derive distribution functions such as given by Eq. (4) or those determined by sampling, as for example, shown in Fig. 7(b). Similar considerations apply to the first return distributions for the periodic pendulum. Again the classical definition of probability seems awkward whereas the frequency definition is more acceptable. Yet even with the frequency definition, the fact that frequencies are accumulated from non-random, periodic occurrences suggests that a probability interpretation is still conceptually problematic. Thus, for the periodic pendulum, probability distributions are formalistic mathematical devices rather than expressions of the likelihood of random events. (We note that the distributions in this paper are calculated using frequency of occurrences.)

The case is different for the chaotic pendulum. Although the state of a deterministic chaotic system is well known in principle, the sensitivity to initial conditions means that an infinite number of digits is required to precisely specify the pendulum's trajectory. Because such specificity is impossible in a laboratory experiment or a computer simulation, the pendulum's angular displacement becomes random. There is now nonzero information entropy and for many systems, the information entropy is a linear function of time.[20]

Chaotic systems therefore have some degree of randomness. Section I alluded to the existence of a hierarchy of randomness in chaotic dynamics.[7] The least random systems are those that obey the ergodic hypothesis; that is, statistical averages over some sort of state space are equivalent to averages in time. Or statistical averages calculated using the classical definition of probability are equal to averages calculated using the frequency definition.[8] This ergodicity is exhibited by the chaotic pendulum. But for the chaotic pendulum, randomness goes beyond ergodicity.

Because the pendulum has friction, its motion in $(\theta, \dot{\theta})$ phase space evolves onto an attractor with a fractal geometry. A small subset of initial points (initial states) on the attractor will evolve through stretching and shrinking of the set such that the individual points will at a much later time move to all parts of the attractor. Because of this mixing of nearby pieces of the attractor, the system is said to be mixing, and the pendulum attractor has this property. A mixing system is more random than an ergodic system.[7]

Further along the randomness scale is a K-system (K for Kolmogorov) for which the information entropy increases linearly. The increase of entropy is related to the characteristic sensitivity to initial conditions. This sensitivity is most easily quantified for a one-dimensional system. Suppose that two points, x_1 and x_2 are initially a distance $\Delta x(0)=|x_1-x_2|$ apart. Then, at some later time, the separation will be $\Delta x(t) = \Delta x(0)e^{\lambda t}$, where λ is the Lyapunov exponent and is positive if the system is chaotic. There are three Lyapunov exponents for the forced, damped pendulum.[21] For chaotic motion, one exponent must be positive in order to provide the characteristic sensitivity to initial conditions and therefore mixing. In this way, the chaotic pendulum meets the criterion for a K-system as well. The pendulum's entropy does increase linearly in time.[20] However, for reasons that are beyond the scope of this paper the chaotic pendulum is not usually considered to be more random than a K-system.[22] Nevertheless, the data of Fig. 9(b) and its comparison with the simple Bernoulli model are intriguing. There may be large regions of the chaotic pendulum's phase space where it behaves like a random two-state system. But whatever its precise degree of randomness, the distributions associated with the chaotic pendulum are reasonably called probability distributions.

VI. SUMMARY

Despite the underlying deterministic nature of the pendulum, there is a noticeable difference in the applicability of probability concepts to regular periodic deterministic systems and to chaotic deterministic systems. Both the classical and frequency definitions of probability are relevant to these differences and both enhance the discussion. Students are familiar with the pendulum and can be introduced to its rich behavior, either by sampling or appropriate laboratory equipment.[23] It is, therefore, an ideal system with which to illustrate and motivate a discussion about determinism and probability.

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[6] See, for example, R. C. Hilborn, *Chaos and Nonlinear Dynamics* (Oxford University Press, Oxford, 1994), p. 14.

[7] E. Ott, *Chaos in Dynamical Systems* (Cambridge U.P., Cambridge, 1993), p. 261.

[8] A. I. Khinchin, *Mathematical Foundations of Statistical Mechanics* (Dover, New York, 1949), p. 52.

[9] In dynamical systems, a Bernoulli system refers to a type of map, the Bernoulli shift map which is tied to coin flipping. In turn, coin flipping is the elementary Bernoulli process that is modeled by the binomial probability distribution.

[10] Return time statistics have been applied to a variety of chaotic systems. See, E. G. Altmann, E. C. da Silva, and I. L. Caldas, "Recurrence time statistics for finite size intervals," CHAOS **14**, 975–981 (2004), and references therein.

[11] R. W. Hamming, *Numerical Methods for Scientists and Engineers*, 2nd ed. (Dover, New York, 1986), p. 132.

[12] In the region between $F \approx 0.617$ and $F \approx 0.659$ the pendulum's motion is

still periodic with the forcing period, but an asymmetry develops. That is, the motion is not symmetric about the vertical but depends on the initial conditions. Both even and odd harmonics of the forcing frequency appear. There are said to be two basins of attraction. If many initial states are used to create the bifurcation diagram, then in the stated region, two branches would occur. However, Fig. 5 was generated using only one initial condition and therefore the pendulum "chose" only one of the asymmetric orbits. Period doubling of this single orbit occurs for the interval $0.658 \leqslant F \leqslant 0.666$ as indicated in Fig. 5.

[13] See, for example, Ref. 6, p. 191.

[14] It is perhaps coincidental but interesting to note that the distribution of long synchronization times for chaotic coupled pendulums may also be modeled by a two-state Bernoulli model. Supporting physical data may be found in H. J. T. Smith, J. A. Blackburn, and G. L. Baker, "Experimental observations of intermittency in coupled chaotic pendulums," Int. J. Bifurcation Chaos Appl. Sci. Eng. **9**, 1907–1916 (1999). The two-state model is presented in G. L. Baker, J. A. Blackburn, and H. J. T. Smith, "A stochastic model of synchronization for chaotic pendulums," Phys. Lett. A **252**, 191–197 (1999).

[15] See Michael C. Mackey, *Times Arrow: The Origins of Thermodynamic Behavior* (Springer, New York, 1991), and J. R. Dorfman, An Introduction to Chaos in Nonequilibrium Statistical Mechanics (Cambridge U. P., Cambridge, 1999).

[16] John Maynard Keynes, *A Treatise on Probability* (Macmillan, London, 1921; Harper and Row, New York, 1962), p. 8.

[17] R. Weatherford, *Philosophical Foundations of Probability Theory* (Routledge and Kegan Paul, London, 1982), p. 5.

[18] See, for example, R. T. Cox, "Probability, frequency, and reasonable expectation," Am. J. Phys. **14**, 1–13 (1946).

[19] Information entropy contains the idea of missing information. The greater the information entropy, the greater the information that is missing for complete specification of the system's state. See L. Brillouin, *Science and Information Theory* (Academic, London, 1962).

[20] H. Atmanspacher and H. Scheingraber, "A fundamental link between system theory and statistical mechanics," Found. Phys. **17**, 939–963 (1987).

[21] Equation (7) can be written as three first-order differential equations in three dynamical variables $\dot{\theta}, \theta$, and $\varphi = \omega_D t$. Thus, the phase space is three-dimensional and requires three Lyapunov exponents to quantify the stretching and shrinking of an initial ball of phase points.

[22] Technically, the pendulum is not hyperbolic because not every point in the phase space possesses distinct directions for stable and unstable manifolds. There are tangencies between the stable and unstable manifolds, and therefore the degree of randomness is limited to that of a K-system; Edward Ott, private communication (2005).

[23] *Deadelon*, *Pasco*, and *TelAtomic* each sell a version of the chaotic pendulum. For a review of these products and further information, see J. A. Blackburn and G. L. Baker, "A comparison of commercial chaotic pendulums," Am. J. Phys. **66**, 821–830 (1998).

Apparatus used by Maxwell to repeat Cavendish's experiment with improved apparatus to verify the inverse-square law of electrostatic attraction. Reference: *The Electrical Researches of the Honourable Henry Cavendish*, edited by J. Clerk Maxwell. (Photo by Jay M. Pasachoff, Williams College, published with permission of the Cavendish Laboratory, Cambridge, and photographed with the assistance of Gordon Squires.)