A comparison of commercial chaotic pendulums

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I. INTRODUCTION

Chaos is an important and fundamental aspect of contemporary nonlinear physics. While numerical simulation of chaos is inexpensive and relatively straightforward, it is difficult and time consuming to construct physical apparatus that actually demonstrates chaos quantitatively. Most institutions must therefore resort to commercially available equipment. Yet these devices are fairly expensive for colleges to purchase and, furthermore, catalog descriptions, by nature, tend to emphasize the strongest features in a given design and to highlight visually pleasing aspects in the illustrations. As a consequence, there remain many questions. What physics does the device actually model? How well does it model the physics? Is it user friendly? Is it sturdy? Can students do a meaningful series of laboratory exercises with the pendulum? For what level is a particular device suited? These are some of the issues we raise in this paper.

We have bench tested three commercially produced chaotic pendulums and report the results as a service to physics educators. While similar in their purposes as experimental chaos platforms, each is based on a slightly different paradigm and in consequence each presents a slightly different window on the chaotic world. Strengths and weaknesses of the design approaches are reviewed here. Since these units are moderately expensive, it is important to choose carefully the pendulum that best suits individual educational and possible research needs.

We are aware of four commercially produced chaotic pendulums. These are manufactured by (in alphabetical order) *Daedalon Corp.*,[1] *Leybold*,[2] *Pasco Scientific*,[3] and *TEL-Atomic, Inc.*[4] We contacted all four manufacturers with invitations to participate in this project—all but *Leybold* ultimately did so.

One author (JAB) brings significant practical experience to the discussion as he is a codesigner[5] of the pendulum manufactured by *Daedalon*. Testing of the apparatus was carried out in his laboratory at Wilfrid Laurier University. For the record, we acknowledge this special circumstance. As an assurance of evenhanded treatment for all participants, each company was given the opportunity to read a preliminary draft of this report and to suggest recommendations to that portion of the manuscript that dealt with their product.

There exist various pendulums and other chaotic devices whose designs have appeared in the literature but which are not commercially available. The recent AJP resource letter on nonlinear dynamics gives references to a whole variety of chaotic devices, many of which have been in this journal.[6] However, this article is directed at those who may have neither the time nor the facilities to construct the type of equipment described in the literature and are therefore interested in making an informed purchase of a pendulum.

For the most part we treat each pendulum separately, focusing first on the fundamental physics, then on hardware aspects of the general design, and finally on software and typical experimental results. It may be noted that all systems are provided with some form of direct computer interfacing for data acquisition. They all have adjustable controls that enable the user to interactively ''find'' various motions. All pendulums are capable of both chaotic and nonchaotic motion. As tested, each one cost over $1500.

II. FUNDAMENTALS

Each commercial package is capable of displaying on a computer screen various graphs that characterize chaotic behavior. In this section we describe the types of displays that are possible with one or more of the pendulums. Further information may be found in sources referenced in the resource letter or from introductory texts.[7,8] Examples of these displays are found in the figures associated with the discussion of each apparatus.

Chaos from a flow (system of differential equations) requires the presence of three dynamical variables; that is, three quantities that are functions of time. For the pendulum, these quantities are the angular displacement of the pendulum θ, the angular velocity $d\theta/dt$, and the phase of the sinusoidal forcing ωt. The first two, θ and $d\theta/dt$, are often represented individually as separate functions of time. When expressed as a sequence of equi-spaced values $\theta(n\Delta\tau)$ or $d\theta(n\Delta\tau)/dt$, these are referred to as *time series*.

The motion of a pendulum may also be graphically represented as an evolving orbit in a space whose coordinates are the dynamical variables themselves, θ, $d\theta/dt$, and ωt; such a figure is known as a *phase portrait*. Since a pendulum can have motion that is at least partially rotary, boundary conditions on the display of the angle θ are made periodic, to obviate the need for an infinitely long axis. Similarly, the forcing phase axis typically has convenient periodic boundary conditions necessitated by the fact that this variable includes time that, in principle, extends to infinity. For a pendulum the periodicity of this axis is usually taken as the forcing period of the pendulum. That is, the time coordinate returns to zero every forcing period. Therefore, a small amplitude periodic pendulum oscillation appears as a single twist of a spiral in the full three-dimensional phase space. For a chaotic pendulum, there are an infinite number of twist-like strands that lead to a complex picture, forming a so-called *strange attractor*. Another common pictorial representation is produced when this three-dimensional phase portrait is projected along the time axis onto a plane and is therefore a *phase plane diagram*. For nonchaotic small amplitude motion the spiral is compressed into an ellipse on the θ, $d\theta/dt$ phase plane. Chaotic motion produces a much more complex phase plane diagram. Although determinism forbids orbits from crossing (in the complete phase space), they may appear to cross in the projection onto the phase plane.

In order to simplify a possibly complex chaotic phase diagram, Henri Poincaré invented a diagram in which the phase portrait is sectioned or cut at some appropriate periodic rate. For the case of a pendulum, the time axis, as noted above, usually has coordinates that are periodic with the forcing. Repetition of sectioning or cuts perpendicular to the time axis—at a particular moment in each forcing period—produces a set of phase points all of which represent instantaneous states of the motion sampled at the same point in successive forcing cycles. Then all such sections are superposed in a single figure. If the resultant figure has just one point, then the motion is exactly periodic with the forcing. If there are two points, the motion repeats itself after two forcing periods and is called *period-2*, and so on. If the motion is chaotic, there are an infinite number of points, again forming a strange attractor whose dimension is one less than the attractor of the complete phase portrait. This section is named, after its inventor, a *Poincaré section*.

The *return map* is a related presentation. In a return map, successive values of a single dynamical variable, such as angular velocity, are displayed. The horizontal axis is used for the nth value of the variable—for example, the angular velocity—and the vertical axis is used for the $(n+k)$ value of the same variable. If there are k points in each forcing period, then the return map gives information that is, in some sense, equivalent to that of a Poincaré section. Typically, a return map is simply first order, meaning that k equals 1, and the values are consecutive in the time series. In this case the interpretation of the map is less straightforward. Devices that use such return maps would need instructor interpretation or at least careful discussion in the accompanying manual. However, when it is not convenient to obtain a Poincaré section, a return map may provide useful information about the motion. The usual application of return maps occurs when the system does not have an obvious built-in periodicity. Different combinations of time series, phase plane diagrams, Poincaré sections, and return maps are produced on computer screens by the various commercial pendulums.

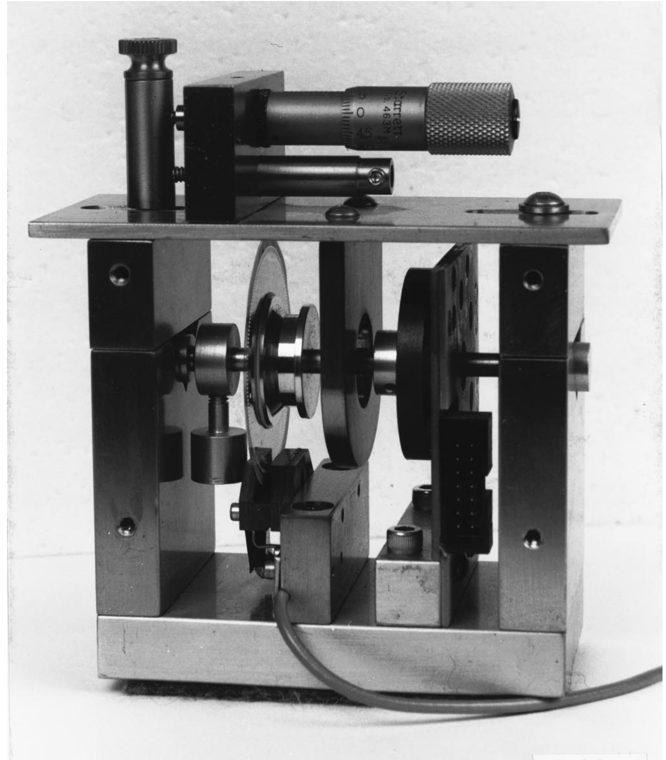
III. THE DAEDALON PENDULUM

A. Physical principles

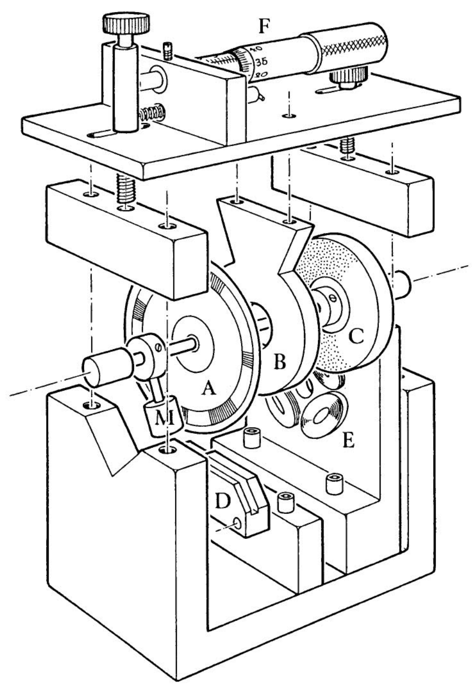
This device is an experimental realization of a simple driven damped pendulum with sine wave applied torque. Its equation of motion is

$$I\frac{d^2\theta}{dt^2}+b\frac{d\theta}{dt}+mgR\sin\theta=A\sin(\omega t), \tag{1}$$

where I is the moment of inertia of all rotating components, b is the friction parameter, m is the mass of the bob, g is the acceleration due to gravity, R is the length of the pendulum, A is the amplitude of the driving torque, and ω is its frequency. Equation (1) is a standard for the damped driven pendulum.[7] For small amplitudes $\sin\theta\approx\theta$ and with this modification, Eq. (1) is readily solvable and the motion has the same form as a damped driven spring. Typical mathematics course treatments of second-order inhomogeneous differential equations discuss at least the linearized version of this equation.[9]



(a)



(b)

Fig. 1. Daedalon: (a) photograph of the pendulum, (b) schematic of the pendulum.

B. Hardware

The *Daedalon* system is comprised of three hardware components: the pendulum itself, a control unit, and a custom PC card. The pendulum is based on a published design.[5] Figure 1(a) and 1(b) is a photograph and schematic diagram of the pendulum and additional illustrations of the apparatus may be found in the *Daedalon* catalog. The pendulum bob is connected via a short rod to a hub on the system axle. At its two ends this axle runs in low-friction bearings which are clamped into V grooves of the support frame. Next to the pendulum and rigidly fixed to the axle is an optical encoder wheel, which is used for reading angular displacement. This thin metal foil disk has 1000 very fine slots etched around its outer rim. A companion sensor unit attached to the apparatus frame directs light through the moving slots, senses the chopped light beam, and converts it into appropriate digital signals which are then sent to the custom PC card. While the encoder wheel has 1000 slots, additional features in the PC card circuitry and software result in a net angular resolution of about ± 0.05 deg.

Farther along is a movable copper plate whose adjustment by the micrometer varies its separation from a ring magnet. The magnet induces both eddy-current damping through its interaction with the copper plate, and torque through its interaction with two pairs of drive coils. Sinusoidal torque is generated via ''a kind of stripped-down brushless slotless linear motor'' (p. 1 of the manual.) This design has the important and necessary property that for any given drive signal, the resulting torque will not depend on the particular angular orientation of the magnet with respect to coils. Thus dc signals give dc torques and ac signals give ac torques.

The interface/control unit contains circuitry which generates a sinusoidal waveform of adjustable frequency and amplitude. A digital display of the driving torque frequency (to within 0.01 Hz) is provided on the front panel. A cable running from this box to the pendulum delivers the torque-producing signal to the drive coils. In addition, a user-supplied arbitrary torque waveform may be substituted for the internal sine wave; a back panel input connector is provided for this purpose. Also on the back panel is an output connector giving access to the drive waveform. This permits a quantitative determination (with the aid of a suitable external scope or meter) of the torque-voltage amplitude, a quantity required for the calibration of the apparatus.

As noted above, damping is mechanically adjusted by means of the micrometer. Repeatability in setting this spacing between the ring magnet and the eddy-current plate allows for precise calibration of the damping coefficient, which appears in Eq. (1).

The third component in the Daedalon system is a custom card which must be installed in the host PC. Data from the optical encoder on the pendulum is passed via a cable to a connector located at the rear of this card. A strobe signal from a rear connector on the interface/control unit is also passed to an input at the rear of the card; this is needed for the generation of Poincaré sections. As described in Ref. 5, autonomous data acquisition is performed by circuitry on the card such that the angle is updated automatically every 7 ms. The host computer can then read these data under suitable software control.

C. Software and experiments

The screen format for the Daedalon-supplied software consists of a relatively large area devoted to the phase plane and a smaller inset portion containing the Poincaré representation. Figure 2 shows the appearance of a typical screen display, which in this case features an oscillating period-2 mode. It is possible to set the horizontal range to any of a specific set of possibilities: $(-\pi,\pi)$, $(-3\pi,3\pi)$, $(-5\pi,5\pi)$, etc. The appropriate factor appears at the bottom

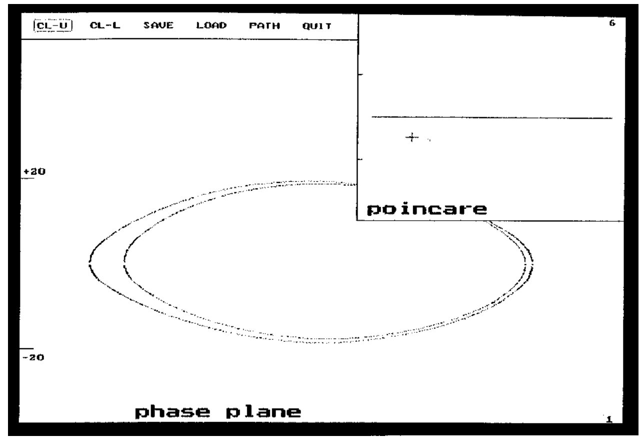


Fig. 2. Daedalon: Screen display of experimental simple oscillating period-2 orbit.

right of the screen. For Fig. 2 this is 1 and so the range is just $(-\pi,\pi)$. A command bar appears across the top of the screen. The CL-U and CL-L commands momentarily clear either of the two windows, the SAVE command initiates a save of up to 5000 data points from either window, the LOAD command retrieves previously saved data from a designated file, and the PATH command specifies the destination (or origin) of data. The files themselves are in ASCII format and consist of rows of three values θ, $d\theta/dt$, and W, where W is the so-called winding number. (The angle θ given in the file is adjusted to be in the range from $-\pi$ to π. The winding number counts the number of multiples of 2π which must be added to the given θ to produce the unfolded angle of the pendulum.) The time for each data triplet may be inferred from the known fixed data interval of 7 ms between readings.

The complexity of even ''simple'' dynamical modes is illustrated by the two different experimentally observed spinning period-2 states shown in Figs. 3 and 4. The integer 3 at the bottom indicates that now the horizontal range is $(-3\pi,3\pi)$. Fully chaotic motion is indicated in Fig. 5. Note the well-developed strange attractor in the Poincaré window.

The laboratory manual that accompanies the Daedalon system describes five experiments in some detail. The first three permit the user to completely determine the physical parameters of the pendulum in terms of the range of settings of the adjustable controls. In these preliminary experiments

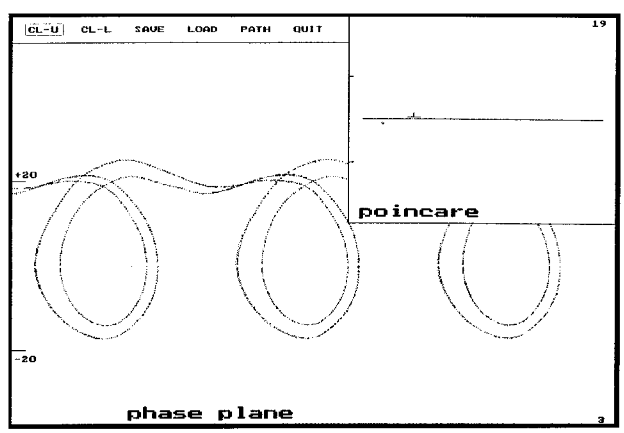


Fig. 3. Daedalon: Screen display of experimental running period-2 orbit.

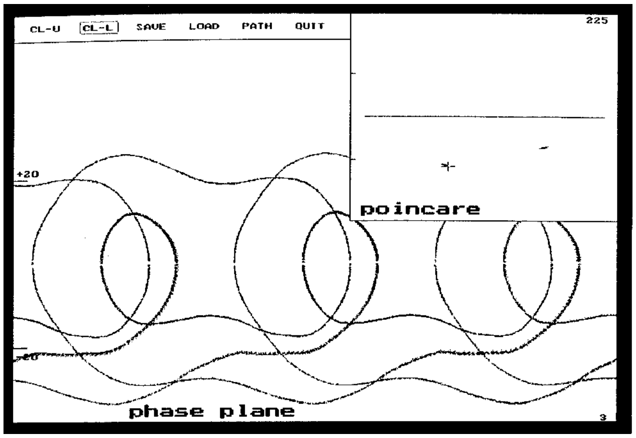


Fig. 4. Daedalon: Screen display of more complex experimental period-2 orbit.

the user measures the *natural* frequency of the pendulum, constructs a graph showing the damping parameter b/I as function of the micrometer setting, and calibrates the torque in terms of an applied voltage. Once the calibrations are completed, the user has a fully characterized chaos tool. These results are essential for subsequent quantitative comparisons of observed experimental data with the numerical simulations of Eq. (1). The write-up for the fourth experiment explains the correspondence of the pendulum with a superconducting device called a Josephson junction with a particular focus on their common property of hysteresis. The fifth experiment studies the resonant response of the pendulum at various values of the damping. The general tone of the manual seems aimed at the upper undergraduate level. The experiments could be done at a lower level but perhaps they would be most useful as part of a special project on chaos or periodic motion.

While the experiments suggested in the laboratory manual are useful exercises in themselves, an important capability of this pendulum is its ability to generate high quality data which may be analyzed with the various algorithms available to chaos aficionados and researchers. For an introductory treatment see Chap. 6 of Ref. 7 and, on a more profound level, the book by Abarbanel.[10] One of the authors (GLB) used acquired time series data from this apparatus to test software for several prediction schemes. The limitation of 5000 points is sometimes problematic but can be somewhat

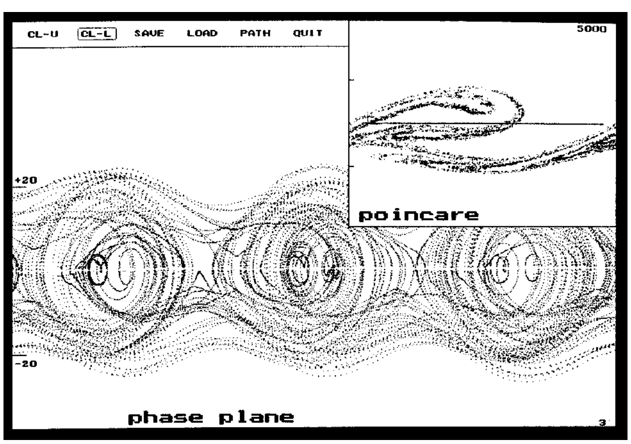


Fig. 5. Daedalon: Screen display of chaotic pendulum motion. Note the strange attractor in the Poincaré inset.

overcome by merging several files in a smooth fashion—the merging needs to be done with user-written software.

The *Daedalon* pendulum originated as a research tool and it can be used for advanced chaos studies. For example, it was employed by various researchers to study control mechanisms for the chaotic pendulum.[11]

D. Device notes

The system is compact and sturdy, well suited for individual use as opposed to lecture-hall demonstration. The setup of the physical pendulum, computer card, and software is quite straightforward. At this writing the price is $2000.

IV. THE PASCO PENDULUM

A. Physical principles

Figure 6(a) shows the essential elements in the Pasco design. A metal disk of radius R is free to rotate in a vertical plane on low friction bearings. On the same shaft is a coaxial pulley of radius r. A string wound on the pulley is attached on either side to linear springs, one of which is then fixed to an anchor point while the other is attached to a drive unit which can move that end of the spring back and forth in a harmonic fashion. A mass m is attached to the large disk at a point which is directly above the suspension point under the condition of zero displacement of the free end of the left-hand spring. Clearly, this inverted point is not one of equilibrium and the mass will flip either to the left or right (as depicted).

The equation of motion for this system is

$$I\,\frac{d^2\theta}{dt^2}+b\,\frac{d\theta}{dt}-mgR\,\sin\,\theta+2kr^2\theta=krA\,\sin(\omega t), \quad (2)$$

where I is the total moment of inertia of all rotating components, b is a velocity-dependent damping constant, m is the added mass, and k is the spring constant. A and ω are the amplitude and frequency of the displacement applied to the left-hand spring. The angle θ is measured from the *upright* vertical position.

As already noted, in contrast to the simple pendulum of Eq. (1) which has a single vertical downward stable equilibrium position, this system has *two* stable equilibrium positions on either side of the vertical—provided that the added mass is sufficient. (For a small mass, there is a single equilibrium point at the upward vertical position.) The effect is similar to that of a double-well oscillator. Possible modes involve oscillations within either the left or right well, or transitions from well to well. However, the configuration of the attached springs prevents the pendulum's mass from going through a complete revolution and therefore the system does not exhibit the quasi-rotary modes of a pendulum whose restoring force is only provided by gravity.

This design and the governing equation bear a close resemblance to a pendulum described by Beckert *et al.*[12] in that it is a combination of a torsion pendulum and simple pendulum.

B. Hardware

The Pasco system consists of four principal components: the disk–mass–pulley assembly, a ''rotary motion sensor,'' which is an optical shaft encoder for monitoring the angular coordinate θ [both of which are visible in the photograph in

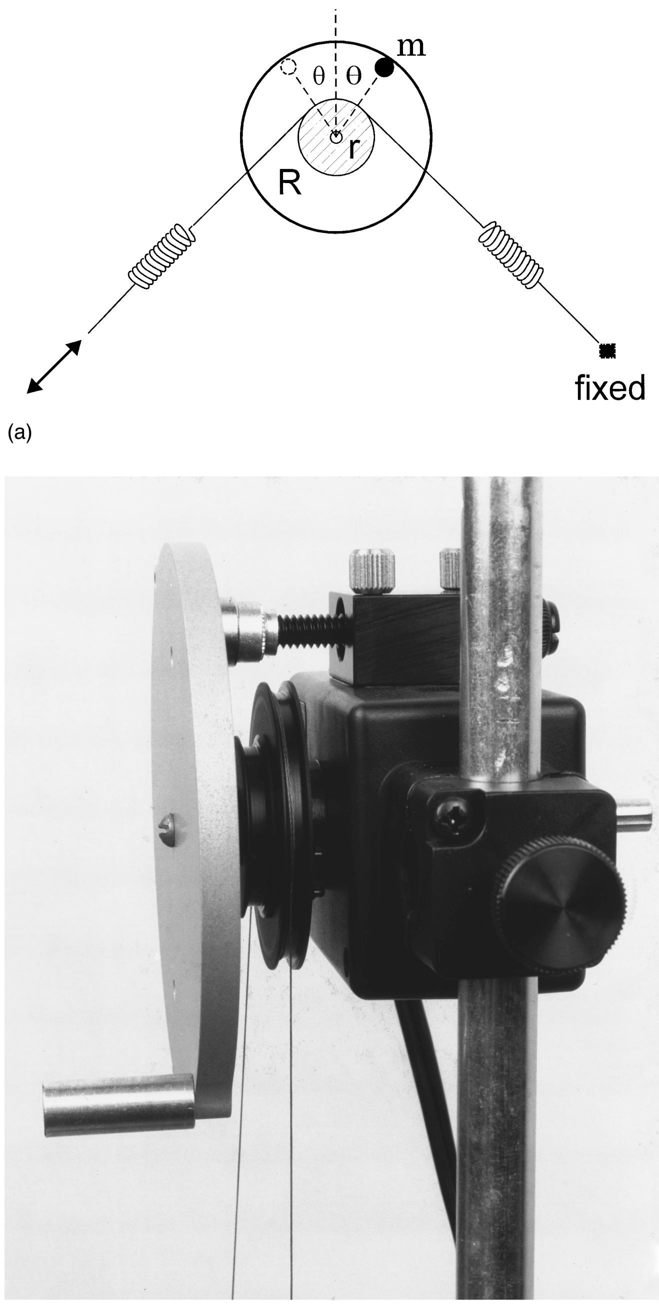


Fig. 6. Pasco: (a) Essential elements of the pendulum including mass m, inertia disk of radius R, pulley of radius r, and two linear springs. The two equilibrium positions are at angles θ on either side of the vertical symmetry axis. (b) Photograph of the disk–mass–pulley assembly.

Fig. 6(b)], a dc motor, which is employed to generate the required harmonic displacements to the free end of the spring, and a model 700 interface box.

The frequency of the forcing term for this system is selected by adjusting the dc voltage supplied to the motor from an external source; the available range is approximately (0.1–1 Hz). The amplitude of this forcing is set by a movable mechanical arm on the motor shaft. Eddy-current damping is set by moving a small permanent magnet toward the rotating metal disk; this component is visible in Fig. 6(b).

The system is interconnected as follows. A cable from the

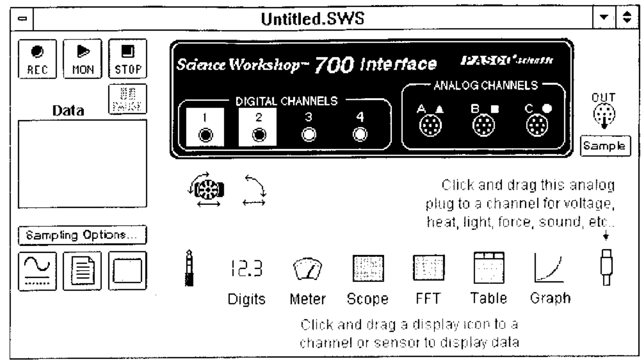


Fig. 7. Pasco: Screen display of the interface unit front panel. The rotary motion sensor has already been installed on digital channels 1 and 2.

rotary motion sensor is plugged into the front of the interface box. The host PC must have an installed SCSI card. A SCSI cable then connects the card to the rear of the interface box. Angular velocity is computed by the software.

C. Software and experiments

Figure 7 shows the initial screen display for the interface box. Activation of the rotary motion sensor is achieved by simply clicking and dragging a plug icon to the desired input channels, and then selecting the correct sensor from a pop-up list. Data display and analysis is also invoked through click-and-drag operations on the screen. Sampling rate is user selectable, but we found 20 Hz to be optimum. This implies that data are acquired at intervals of 50 ms.

Figure 8 shows a period-5 orbit obtained with the apparatus. This complex but regular motion involves oscillations within and transfers between the two potential wells. A phase plane depiction of chaotic behavior is shown in Fig. 9; here the transfers between wells are irregular and unpredictable. Portions of the corresponding time series and data table for this chaotic state are combined in the screen display of Fig. 10. There is no provision for a Poincaré section or a return map.

Experiments suitable for this apparatus are described in the *Workshop Physics Activity Guide*, a comprehensive manual for introductory physics experiments written by Priscilla Laws.[13] The complete *Guide* may be purchased or

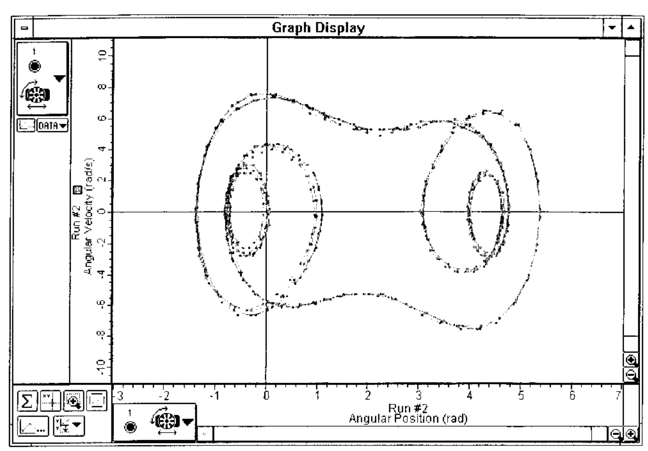


Fig. 8. Pasco: Experimentally observed period-5 orbit involving transfers between the two potential wells.

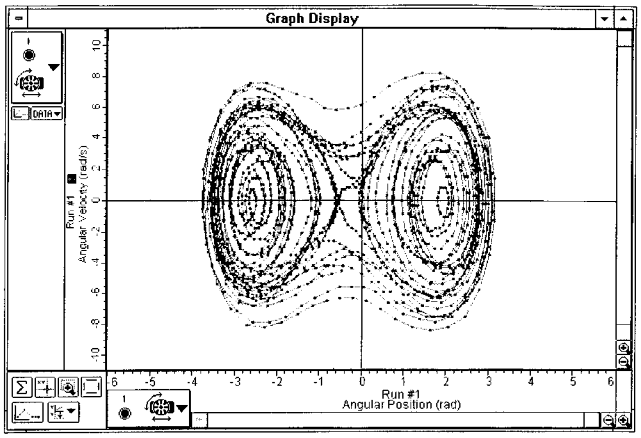


Fig. 9. Pasco: Experimentally observed chaotic motion involving motions within each potential well and transfers between them.

just Module 2, of which roughly 17 pages are devoted to a variety of exercises that are designed to help the student understand the forces delineated in model equation (2). They also give some sense of the important physical parameters in oscillatory motion, and plot phase plane and time series graphs. There is a nice exercise, using time series, that illustrates the chaotic property of sensitivity to initial conditions. While the *Guide* provided detailed instructions for ten activities, we were unable to find model equation (2) anywhere. Perhaps this reflects the generally introductory nature of the *Guide*.

Although not featured in the *Guide*, we think that more should be done with the possibilities of calibrating the system. It would seem straightforward to measure the mass m, the radius of pulley r, the radius of the wheel R, and the radius of the motor arm A. The spring constant could be determined by noting that Eq. (2) yields for the static case (zero time derivatives and no forcing)

$$k=\frac{mgR}{2r^2}\,\frac{\sin\,\theta_e}{\theta_e}. \tag{3}$$

Hence measuring the equilibrium angular displacement θ_e gives k. The forcing frequency ω could be measured if a second transducer were attached to the driver. Finally, one would like to make a calibration curve for the damping b as a function of the distance of the magnet from the disk. This could be achieved by either of two methods—similar to those suggested in the *Daedalon* manual. (a) With no drive

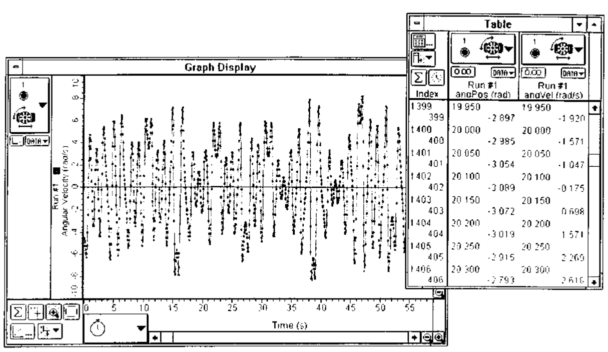


Fig. 10. Pasco: Portion of experimental chaotic data displayed as a time series and in table format.

on, displace the system slightly from equilibrium and let the pendulum oscillate to a resting position. Fit the acquired data with an exponentially decaying sinusoid which is the analytic solution to the small amplitude equation of motion. Doing this for several values of the magnet's distance from the disk should provide a calibration curve for b. (b) Remove the spring and drive from the disk. Set the disk on its side so that gravity is removed from the system. Now the equation of motion is simply

$$I\,\frac{d^2\theta}{dt^2}+b\,\frac{d\theta}{dt}=0. \tag{4}$$

If the system is given an initial push the angular velocity decays as

$$\frac{d\theta}{dt}=\left(\frac{d\theta}{dt}\right)_0 e^{-(b/I)t}. \tag{5}$$

This curve is simpler than that developed in method (a) and can serve as the basis for a calibration curve of b vs distance.

D. Device notes

The *Pasco* pendulum seems to be well built and robust. It is important to keep the string wrapped tightly around the pulley so that there is no slippage. One may achieve this by threading a loop of the line into an existing small hole in the pulley rim and securing it with a cross wire. Data files can have lengths that exceed 25 000 points depending on the configuration used. In the high resolution mode the angular data are accurate to about ± 0.25 deg.

The computer interfacing requires the user to have a SCSI interface card installed in the PC. This is in contrast to the *Tel-Atomic* apparatus, which uses a serial port, and the *Daedalon* apparatus, which uses a supplied custom interface. The interface box provides an old-style SCSI connector plug and it should be noted that many SCSI cards use a newer high density connector plug.

The screen graphics are good and the data can be displayed in tabular format or transferred to a spreadsheet for analysis. Scales are given for the axes of all figures. However, positioning of the angular displacement scale seems to depend on the position of the device at the time it is started.

The equipment and software setup is fairly straightforward, except for knowing little tricks such as how to keep the string tight around the pulley. Finally, aside from the *Guide* there was no other manual so setup relies on the instructor's native ingenuity. The complete *Pasco* system covers many other experiments which, however, share various common components—including the interface box. The system has been designed to be both Windows and Mac compatible. Thus many of the parts required for this chaos experiment can, if desired, serve in other applications within the *Pasco* repertoire. Some items, such as the low voltage power supply may already be on the user's shelf. However, the cost for all the parts necessary to do the experiments is about $1600.

E. Note added after manufacturer's review

In consideration of the manufacturer's review of this manuscript, we add the following remarks. Our tests were done using the ''700'' interface box supplied to us and pictured in Fig. 7. The manufacturer notes that chaos experiments can also be performed with a less expensive ''500''

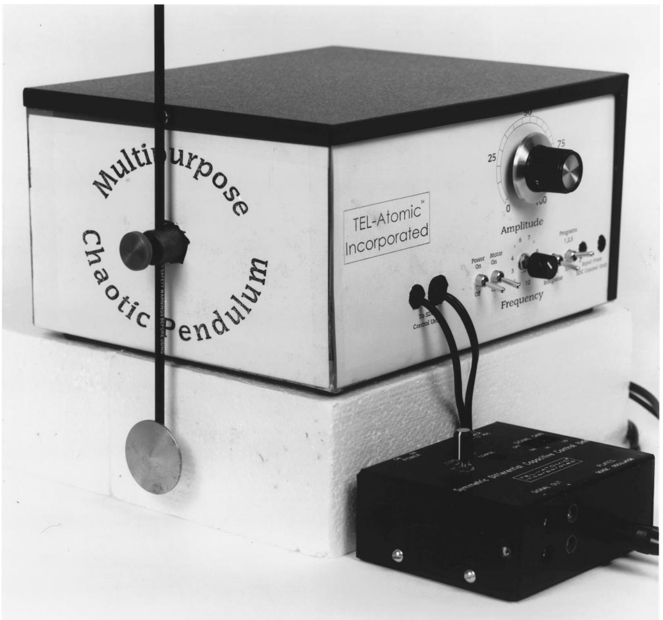


Fig. 11. TEL-Atomic: Photograph of the system including the pendulum, main drive and control box, and small SDC control unit.

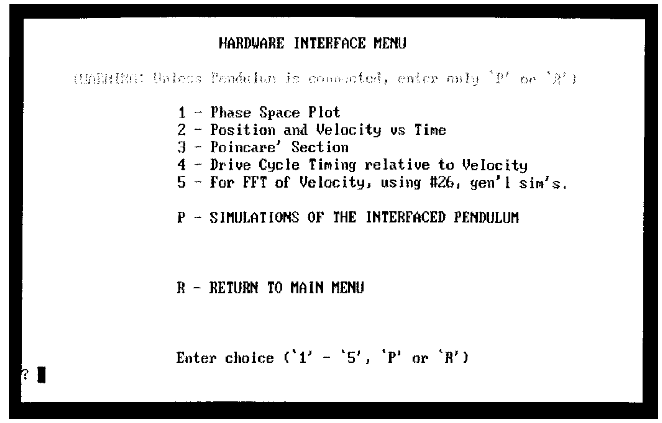


Fig. 12. TEL-Atomic: Screen display of the opening hardware interface menu.

interface using the serial port, thus obviating the need for a SCSI card. However, the 500 interface has only two digital inputs as needed for a single rotary sensor. We used an extra set of ports on the 700 to measure the frequency of the driver as well. Therefore use of the 500 box would require finding another way to measure the driver frequency. However, the saving is considerable as the 500 interface is about $400 less than the 700 interface box.

V. THE TEL-ATOMIC PENDULUM

A. Physical principles

A photograph of the *TEL-Atomic* pendulum is shown in Fig. 11. The equation of motion for this pendulum is similar to that of the *Daedalon* pendulum except that the forcing is provided by a square wave rather than a sinusoidal torque. The appropriate equation is

$$I\,\frac{d^2\theta}{dt^2}+b\,\frac{d\theta}{dt}+mgR\,\sin\,\theta=\tau_{\pm}(t), \tag{6}$$

where the parameters have the usual meanings and $\tau_{\pm}(t)$ refers to the square wave drive. (In this case, m is total mass and R is an effective radius that includes positioning of the bob and the arm of the pendulum.) We know of no published simulations of pendulums that are driven by square wave torques so this design is unusual in that respect. However, the motion is similar to a sine wave drive, in part because, as indicated below, induction motors do not have constant torque and therefore the drive is not likely to be a perfect square wave. Consequently, the various figures produced by this device are similar to those one finds for a sinusoidally driven system.

B. Hardware

The system consists of two units: a large box from which emerges a shaft which serves as the pendulum axle, and a small so-called SDC control unit. As indicated in Fig. 11, the *TEL-Atomic* pendulum has the most obvious ''look'' of a pendulum. The pendulum arm slides through a hole in the shaft and thus can have a length ranging from 1 to 9 in. depending on where the screw clamp is tightened. The pendulum axle is in fact the shaft of an induction motor that is housed inside the main module box.

The mechanism for sensing the pendulum motion is unique. Two permanent magnets are mounted on a u-shaped yoke such that they are positioned on either side of an aluminum disk that is mounted on the motor shaft. The yoke is itself fixed to a beryllium–copper cantilever beam. As the pendulum rotates, so does the aluminum disk. Differential motion between the disk and the nearly stationary magnets induces eddy currents which in turn slightly deflect the cantilever beam. As the beam flexes, a small metal plate or vane is caused to move between the plates of a capacitor. This capacitor forms part of the bridge arrangement called a ''symmetric differential capacitor sensor'' (SDC). A special purpose integrated circuit (NE5521: LVDT signal conditioner) located in the separate SDC control unit performs the bridge demodulation. Since the eddy current drag is proportional to the velocity of the metal disk past the magnets, the net effect is to measure the pendulum angular velocity (as opposed to the angle measurement of the Daedalon and Pasco pendulums). The angular coordinate is then derived electronically from the velocity by an op-amp integrator.

As mentioned above the applied torque is approximately a square wave rather than the usual sine wave. It is generated in the following manner. Standard 60-Hz ac power is supplied to the induction motor coils—one of which contains a

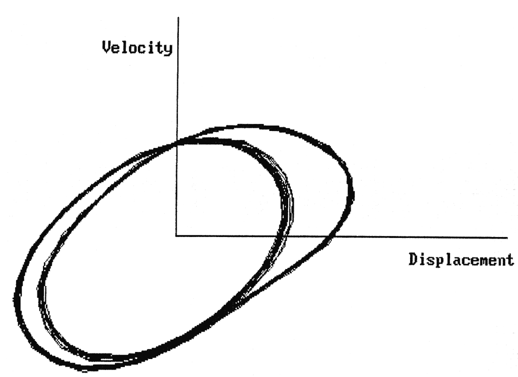


Fig. 13. TEL-Atomic: Screen display of experimental period-2 motion.

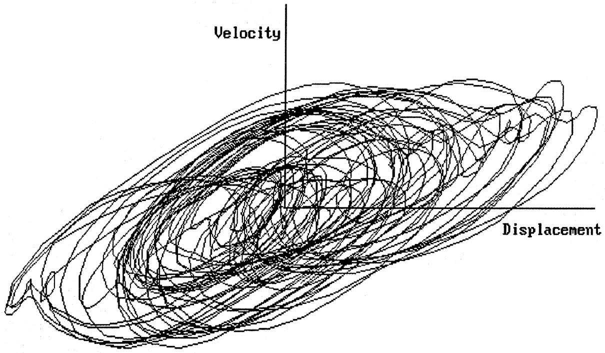


Fig. 14. TEL-Atomic: Screen display of chaotic motion for the pendulum.

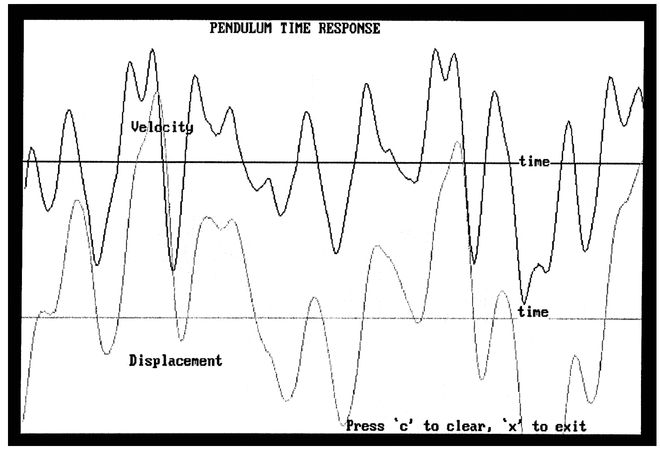


Fig. 15. TEL-Atomic: Screen display in Mode#2—position and velocity vs time, in this case for chaotic motion.

phase-shifting capacitor. The location of this capacitor determines the direction of rotation—clockwise (CW) or counterclockwise (CCW). A 555 timer running at 50% duty cycle drives a relay which is used to alternately swap the capacitor between the coils and therefore to alternate the torque direction back and forth between CW and CCW. The *magnitude* of the torque is controlled by a variable resistor that is wired in series with the motor windings and thus sets the ac current. External controls for the 555 frequency and the variable resistor are provided on the front panel of the main unit.

It should be noted that the applied torque cannot match the assumed ideal square waveform for two reasons. First, induction motors do not actually generate constant torque,[14] so there may be some residual ripple. Second, there will be deviations from quasi-steady torque at times when the shaft angular velocity (pendulum ω) is near zero.

The main unit has two controls, one for the adjustment of drive amplitude and the other for adjustment of frequency. Because the aluminum disk and magnets have a fixed geometry, the eddy current damping is also fixed. This means that the parameter b cannot be adjusted independently. However, by changing the length of the pendulum itself, the ratio b/I can be altered. This, of course, also affects the natural frequency of the pendulum.

C. Software and experiments

The software is DOS based and the initial menu provides two main options: *hardware* and *simulations*. The hardware option shown in Fig. 12 supports the physical pendulum and provides the kinds of graphics shown in previous figures. The simulation menu lists over 40 simulations ranging—alphabetically—from ''autocorrelation'' to ''zany root finder,'' and includes many of the popular chaotic systems. One purpose of the simulations is to provide an introduction to the kind of behavior exhibited by the ''real'' pendulum. While the Tel-Atomic simulation software is unique, many commercial packages dealing with the pendulum are also available.[15] However, the purpose of this article is a comparison of real pendulums and therefore we do not discuss the simulation software further.

Figures 13 shows a screen display of the phase plane for a case of simple period-2 motion. Note the similarity to the orbit shown for a sinusoidally driven system in Fig. 2. Note also that the coordinate axes appear to be rotated from their expected orientation. This deformation is an artifact of the phase shift introduced by a ''leaky'' integrator used to electronically generate the phase coordinate θ from the sensor signal which is proportional to angular velocity $d\theta/dt$.

Figure 14 shows a chaotic state of the pendulum. Figure 15 shows the simultaneous display of portions of time series of angular displacement and angular velocity for a chaotic state. Figure 16 shows a display that is labeled as a Poincaré section, but is really a return map for a chaotic pendulum, as described in the introductory discussion above. The axes are labeled although numerical scaling is not provided.

The laboratory manual that accompanies the *TEL-Atomic* pendulum is large, over 140 pages, and consists of three parts. The first part describes the physical pendulum and suggests, in broad terms, the kinds of exercises one might do with the pendulum. The second part describes the simulations, and the third part is a series of appendices to the earlier sections.

The manual is unconventional both in style and content. Its style is that of a conversation or tutorial with its author, Randall Peters of Texas Technical University. The contents range broadly over the field of mechanical pendulums, a field to which Dr. Peters is a contributor. Particular attention is given to the SDC sensing mechanism. Experiments are suggested in outline rather than prescribed in detail. Calibration of relevant physical parameters by the author is summarized complete with numerical values, but the details of the methodology are left to the reader. It is important that the instructor be sufficiently knowledgeable to guide students through

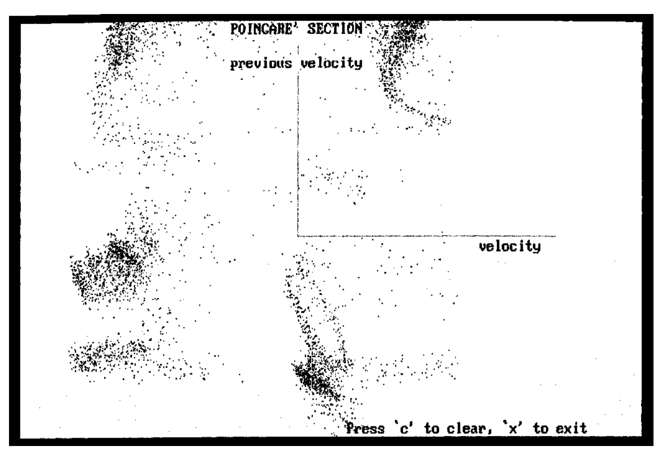


Fig. 16. TEL-Atomic: Screen display of chaotic pendulum motion in Mode#3—Poincaré section.

Table I. Summary of chaotic apparatus.

Type	Daedalon Pendulum	Pasco Double-well oscillator	TEL-Atomic Pendulum
Excitation	Sinusoidal	Sinusoidal	Square wave
Setup	Some effort	Some effort	Simple
Construction	Precision	Very good	Good
Physical Size	Compact	Medium	Medium
Manual supplied	Yes	Can be purchased	Yes
Expt. descriptions	5 expts.	10 ''activities''	Suggestions
Forcing freq. adj.	Yes	Yes	Yes
Forcing ampl. adj.	Yes	Yes	Yes
Damping adj.	Yes	Yes	Indirectly through I
Calibrate	Yes	With modification	To some extent
Quantitative meas.	Yes	Possible	Mostly qualitative
Display time series	With user software	Yes	Yes
Phase plane	Yes	Yes	Yes
Poincaré section	Yes	No	No
Return map	With user software	With user software	Yes
Data acquired	Angle	Angle	Angular velocity
Acquisition method	Optical encoder	Optical encoder	SDC sensor
Data files	Yes	Yes	No
# points in data file	5000	$>$25 000	NA
Sample time interval	7 ms	Optimum at 50 ms	20 ms
Interface through	Supplied card	Needs SCSI card	Serial port
Software	Dedicated, DOS	Multipurpose, Windows or Mac	Dedicated, DOS
Parts	All dedicated	Multipurpose	SDC interface is multipurpose
Price	$2000	About $1600 as tested, or $1200 with ''500'' Interface	$1605

the necessary steps to carry out the suggested experiments. Since data cannot be saved in a file, there is no ability to do analysis. Calibration of, for example, the damping constant by a curve fit is not possible. Nor can one look at data from a single run in a variety of displays. This is a nontrivial limitation because experimental chaotic data are sensitive to initial conditions and therefore not repeatable. Thus, this apparatus is, in contrast to the others, essentially *qualitative*.

From the viewpoint of physics pedagogy the important feature of the *TEL-Atomic* pendulum is its ability to display on screen (a) phase plane diagrams, (b) time series for angular position and velocity, and (c) return maps. The pendulum is comparatively large and hence more easily seen from a distance than either the Daedalon or Pasco pendulums. Therefore it seems best suited to the lecture/demonstration format for a medium-sized class.

D. Device notes

The approach to motion sensing in this design is rather unique. Data acquisition occurs via the SDC control unit. The central feature is a mechanically variable capacitor which is monitored within a capacitor bridge. The resulting bridge offset voltage is proportional to the pendulum's angular velocity. The angle itself is generated by an electronic analog integrator. Data are acquired at the rate of 50 Hz (20-ms intervals) and are transferred to the host PC through a serial port.

The *TEL-Atomic* pendulum with the SDC unit required for computer display costs $1605.

E. Note added after manufacturer's review

In consideration of the manufacturer's review of this manuscript, we add the following remarks. We noted above that the system as tested has no capability for file saving. The system designer says that this can be remedied fairly easily and that future versions of the device will have a file-saving capability. We also noted that the system does use approximate square wave forcing rather than sinusoidal. However, the difference between the *results* of these two types of waveforms may be less than initially expected. The dominant Fourier component in the square wave will be the sinusoidal fundamental: Higher frequency harmonics are smaller in amplitude and also probably less important for the dynamics in the parameter space that is of interest; that is, low frequency forcing. In some sense the dynamics are not especially sensitive to the higher frequency components.

VI. CONCLUSION

These pendulums are all complex devices. They approach the experimental task of exploring chaotic dynamics in unique and different ways. We present a summary of their salient points in Table I. The final choice of a particular apparatus must depend on a number of factors, including projected use, price, and student level.

Finally, we would like to express our appreciation to all the manufacturers for their friendly cooperation and patience in making equipment available for the purposes of this article.

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[1]Daedalon Corp., Box 2028, Salem, MA.

[2]Leybold Didactic GMBH, D-50354 Huerth, Germany.

[3]Pasco Scientific, 10101 Foothills Blvd., Roseville, CA.

[4]TEL-Atomic, Inc., Box 924, Jackson, MI.

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[14]B. I. Bleaney and B. Bleaney, *Electricity and Magnetism* (Oxford U. P., London, 1959), see pp. 319–321 for a discussion of torque in ac induction motors.

[15]Here are three examples. Page 256 of Ref. 7 provides order information for inexpensive software that is primarily but not exclusively dedicated to the pendulum. More comprehensive intermediate-level general chaos packages are available from Physics Academic Software, Box 8202, North Carolina State University, Raleigh, NC 27695-8202. Advanced level software is available with the book by H. E. Nusse and J. A. Yorke, *Dynamics: Numerical Explorations* (Springer-Verlag, New York, 1994).