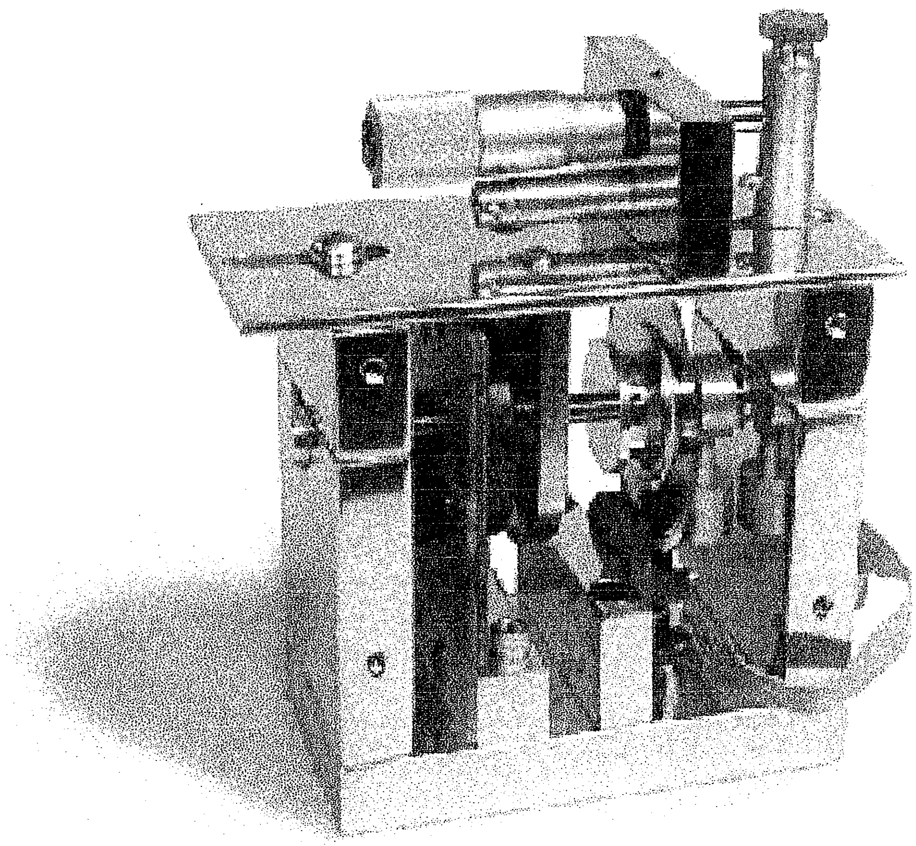
Chaotic Pendulum User's Guide



Contents

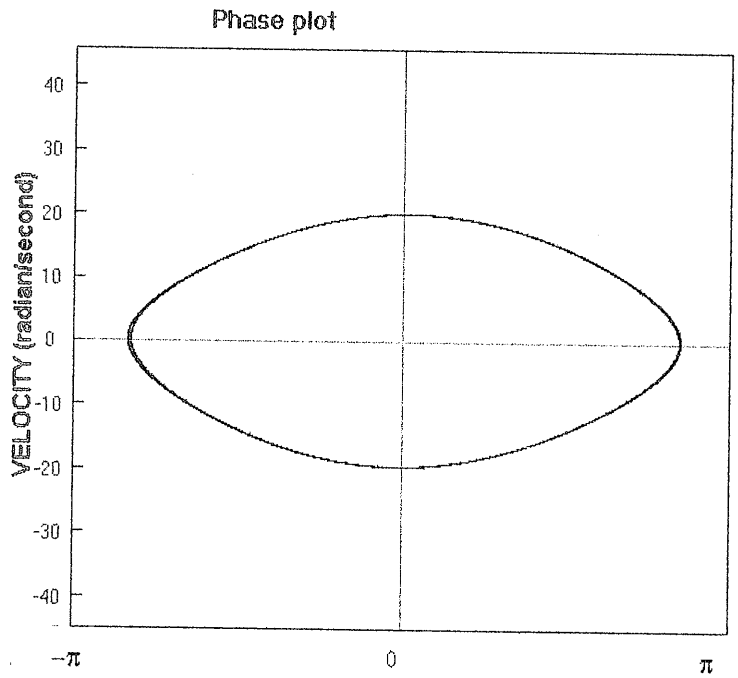
1. Introduction

Chaos is familiar to all of us. Some mechanical, biological, electronic, social, astronomical systems that exhibit chaos are well defined and have been extensively studied. A key feature of a chaotic system is that future behavior cannot be predicted even though the system can be fully described mathematically and its past behavior is completely known. In recent years, chaos has become an active academic discipline that spans many branches of the sciences and engineering. The damped driven pendulum is an example of a chaotic system that can be readily formulated mathematically and can also be reproduced in a laboratory. For these reason it is a good tool for teaching the fundamental properties of chaos. Systems that exhibit chaos may also exhibit resonance, hysteresis, periodic and multi-periodic motion. The damped driven pendulum can show all of these phenomena.

The Daedalon pendulum is mounted on a shaft that runs in miniature low friction ball bearings. A torque converter is mounted on the same shaft as the pendulum. The torque converter is a brushless, slotless, linear DC motor of the type that is found in many tape recorders and it applies a torque that is proportional to the voltage that is applied to the torque converter. In this application the torque converter is connected to an ac source in the electronic pendulum driver supply. An eddy current damping ring, which is close to the permanent magnet of the torque converter provides a damping torque, which can be calibrated. A digitizing wheel which has 1000 small slots close to its circumference and which is mounted on the same shaft as the pendulum and the torque converter measures the angular position of the pendulum. The slots of the digitizing wheel pass through the gap of an optical encoder. On one side of the gap is an LED light source. On the other side are a phase plate and two optical detectors. The phase plate is a foil located in front of the optical detectors and it has two adjacent sets of stationary slots that are the same width as those of the digitizing wheel. The two sets of slots are offset from each other by a half a slot width. A photo detector is behind each set of slots. The photo detectors generate two sets of electric pulses each time a slot of the digitizing wheel passes through the optical encoder. The second set of pulses is termed the quadrature signal. These two series of pulses are offset in time by a half pulse width. The two series of pulse are connected to an electronic interface that calculates the angular position of the pendulum and the direction of rotation. The electronic interface uses an internal clock to calculate the angular velocity of the pendulum from the angular position and the direction of rotation. A timing pulse is generated at the start of each electronic drive cycle and it is sent via a BNC cable to the electronic interface. The two standard methods of viewing dynamic information are the phase plane plot and the Poincaré plot.

A phase plane plot has angular position plotted horizontally and angular velocity plotted vertically. Every data point that is generated is plotted. Figure 1.1 shows periodic motion, which is an ellipse. Multi-periodic motion will be a more complex figure that switches between two or more distorted ellipses. Chaotic motion is manifest as many merging ellipses, and the figure may be so complex that it is not possible to extract useful information. Chapter 6 discusses multi-periodic and chaotic motion and fig. 6.1 and fig. 6.4 are examples of multi-periodic and chaotic phase plane plots.

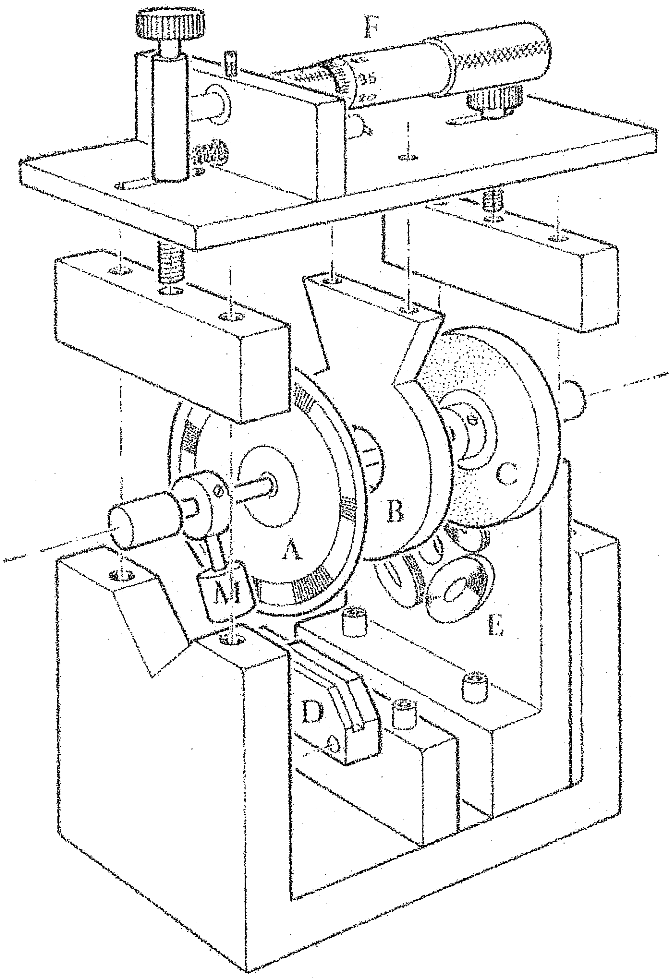
Figure 1.1



The phase plot for chaotic motion can be complex. For this reason a subset of that plot, a Poincaré plot is also used. It is a plot of angular velocity versus angular position but with only one data point per cycle that is determined by the timing pulse of the electronic drive oscillator of the ac source. The phase plane plot is useful for identifying periodic modes but is not helpful with understanding chaotic modes whereas the Poincaré plot is useful for to studying chaotic modes. The software includes calculation of capacity dimension, Lyapunov exponent, and power spectra, as well as numerical simulation procedures. The complete system is suitable for use in an undergraduate laboratory.

Description of the pendulum

Mechanical



The apparatus has a shaft that runs between two low friction ball bearings in cylindrical carriers that are clamped into vee grooves of the support frame. The shaft carries the pendulum, which is a short rod with a mass (M) on the end, a ring magnet (C) and a slotted wheel (A). The ring magnet, which has eight poles, has two purposes. It rotates adjacent to a stationary copper ring. Eddy currents that are generated in the copper ring provide a damping torque to the motion of the axle. The magnitude of the damping torque can be adjusted by varying the distance between the magnet and the copper ring. The micrometer screw on the top of the frame sets the position of the copper ring. The ring magnet also forms part of the brushless slotless linear motor. This enables a known torque to be applied to the axle. The two pair of coils (E) is the other part of the motor. The slotted wheel has a 1000 slots and it runs between the LED source and the optical detector of the optical encoder (D).

USB interface

The electronic interface circuit has a microprocessor and a USB (Universal Serial Bus) engine. The data from the optical encoder and the timing pulse are directed to a port of the microprocessor which is programmed to calculate the angular position and the angular velocity and it converts them and the timing pulse to the USB format. The microprocessor transmits this information to the USB engine, which is programmed to communicate with a USB port of a PC via a USB cable. The remainder of the data processing is carried out in the PC.

2. Software

Installing the software

Software is supplied for PC's that are using either Windows XP or Windows 2000 operating systems. Installation of the software should only take a few minutes.

1. Close all of the existing applications.
2. Do not connect the interface electronics to a USB port at this time.
3. Find the letter of your CD drive.
4. Put the Daedalon CD into the drive.
5. If it does not start automatically, then go to START\RUN and enter D:\setup.exe where D is the letter of your CD drive. Hit OK.
6. It will install eight bootstrap '.dll' files. If any of these files already exists on your computer and your version of the file is later than the one that is about to be installed then a panel will alert you of this conflict and you should choose the option to keep your original file. Do not install the earlier file. A "Welcome to the installation program" panel will appear after the bootstrap files are installed. It will say that you cannot install shared files if they are in use. If you have not already closed all other applications then exit the install program and close all other applications and start again. Otherwise click on OK.
7. At the next panel click on the left most button in order to install the pendulum software.
8. The next panel should indicate that the program group is *Daedalon*. Click on continue.
9. A final panel indicating that the software has been successfully installed will appear. The executable and supporting files will be in the folder c:\Program files\Daedalon\. Click on OK.

10. There is one further step. The driver file must be installed in the windows driver registry. Connect one end of the USB cable to the Daedalon interface box and the other end to a USB port of your PC.
11. Wait a few seconds and a message will appear that indicates that new hardware has been detected.
12. A panel that is entitled "Welcome to the Found new Hardware Wizard" will appear. With the CD still in the drive click on NEXT.
13. A panel will appear that says that the software that you are installing has not passed Windows Logo testing. Click on continue anyway.
14. The "Completing the Found new Hardware Wizard" will appear. Click on Finish. The drivers are now registered and the software is ready to use.

NOTE: What the LED's show

The interface box has a microprocessor that collects data from the pendulum and transmits it to the PC. However, the interface box does not have any program code permanently stored in it. Each time that the interface box is connected to the PC, software is downloaded from the PC to the microprocessor in the interface box. When the USB plug from the interface box is connected to the PC a signal automatically is sent to the PC and an initial Driver program for the interface box is located. During this process, the LED shows RED. This initial driver program then signals the PC and initiates the download of a second driver program to the interface box. This second driver program contains the microprocessor software. This program starts running as soon as it has been loaded. At this point the LED shows GREEN and the pendulum data starts to be sent to the PC. The complete process may take a few seconds. The Daedalon program may be started at any time after the LED shows GREEN.

Operating the software

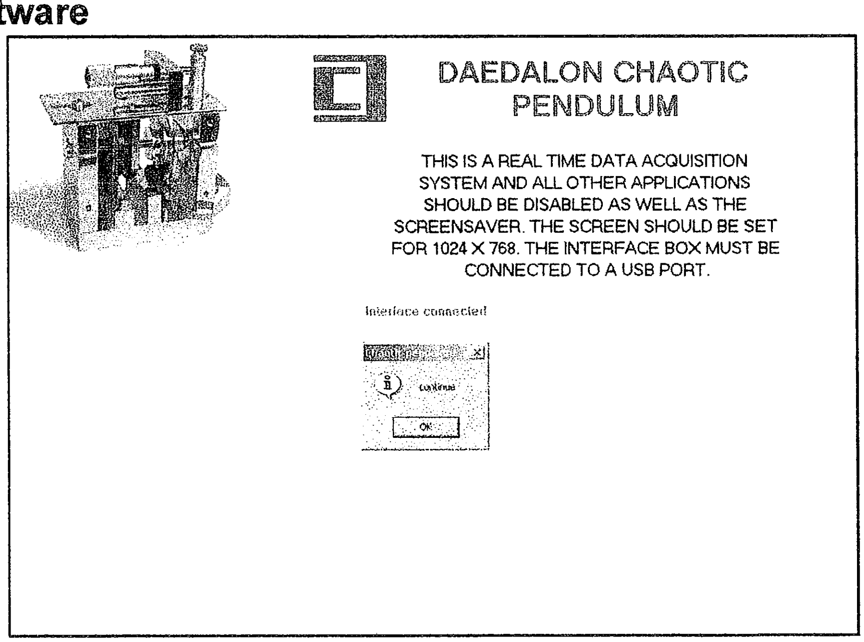
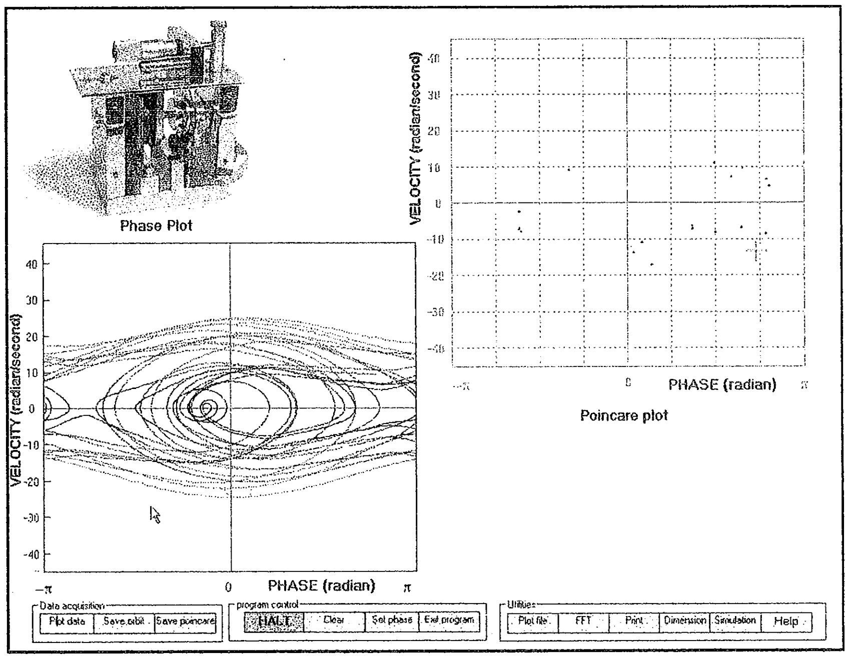


Figure 2.1

As the Daedalon software collects data in real time it is necessary to close all other applications and the screen saver (START\Settings\Control Panel\Display\Screen Saver\None).

When the Daedalon program is started it will open with a message (Figure 2.1) that the interface is or is not connected. If the interface box has not been connected you may quit the program at this point or you may proceed and use any of the program utilities that do not require the pendulum and the interface circuit. If the interface has been connected then click yes and continue to the main window (Figure 2.2).

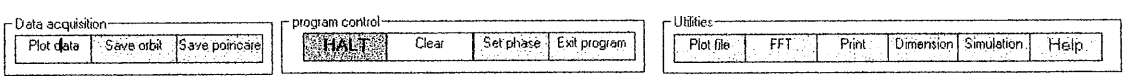
Figure 2.2

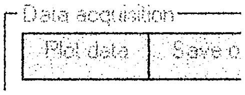


If the interface is connected, all thirteen buttons of the menu will be usable but if the interface is not connected then the three menu buttons that control the acquisition of data from the experimental pendulum will not be visible but the ten remaining buttons that control the utilities will be active.

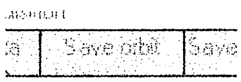
What the buttons do

The main menu bar looks like this



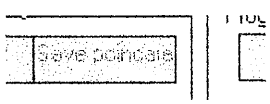


Plot data will record data from the experimental pendulum and plot the data on the screen to a Phase plot and to a Poincaré plot. Clicking on the GREEN HALT button will stop data recording.

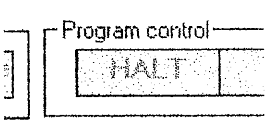


Save Orbit will record data from the experimental pendulum and plot the data to a phase plot on the screen and it will save <u>data to a file</u>. You will be asked to enter the number of phase data points and a file name. The default is 1000 phase points. The pendulum generates 500 phase data points per second. The maximum number of phase data points that can be stored in a file is 4000000. The session may be halted at any time by clicking on the RED HALT button that will change to gray at the completion of recording data and the file will be deleted.

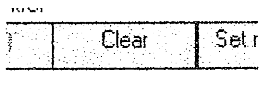
Data to a file is recorded in an ASCII format. The first line of the file reads either PHASE or POINCARE. The second line of the file is the number of data points. The data follows with one data point per line. The first item on a line of data is the angle (radians) in the range [$-\pi<$angle$<\pi$]. The second item is the angular velocity (rad/second) and in the case of PHASE data the third item is the number of times (integer) that the pendulum has rolled over since the data started to be recorded. The POINCARE data file does not record the rollover. The three items are separated by single spaces.



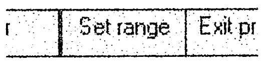
Save Poincaré will record data from the experimental pendulum and plot a Poincaré plot on the screen. The data is also saved on a <u>file</u>. You will be asked to enter the number of Poincaré data points and a file name. The Poincaré data points are generated at the same rate as the drive frequency – about one point every second. The session may be halted at any time by clicking on the RED HALT button and the file will be deleted. The red HALT button will switch to gray at completion of the recording.



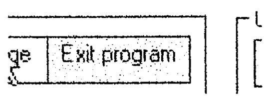
HALT will stop the recording of data under any circumstance. The HALT button will be green while plotting experimental or simulated data to the screen and it will be red while saving experimental or simulated data to a file. The HALT button is yellow during calculation of dimension. The HALT button is magenta while the FFT and the Lyapunov exponent are being calculated.



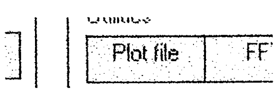
Clear will erase the phase plot screen during the recording of data from either the experimental pendulum or from the simulation. It will not halt the recording of data but only data from this point forward will be recorded.



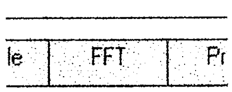
Set range enables the horizontal scale of the screen phase plot to be changed by a slide bar from {–π to π} to one of {–nπ to nπ} where n is an integer 3, 5, 7, 17, or 33. This will not affect the saving of phase data to a file or a Poincaré plot. So-called chaotic diffusion can be observed by setting n to a large value.



Exit will terminate the program and return to the windows desktop as long as the software is not collecting or processing data.



Plot file enables either previously recorded phase data or Poincaré or power spectra data from a file to be plotted on the screen. Set phase may be used to change the horizontal scale of the phase plot. You will be asked to enter the file name. But you do not have to enter the extension. Each data file has a header that indicates whether it is a set of Phase plot data, a set of Poincaré data or FFT data. Clicking on the YELLOW HALT button will halt the session.



FFT is a Cooley-Tukey fast Fourier routine that generates a Power spectrum from a time series of velocity data in a file. The FFT uses 2048 data points that are spaced 0.036 seconds apart. The data acquisition takes readings spaced 0.002 seconds apart. Thus the FFT uses only every eighteenth data point. The points in the spectra are 0.08522 r/s apart. You will then be asked to select a "window".

Parzen
Hanning
Welch
Square

Select an FFT window

An FFT can only analyze a finite number of data points. A correct spectrum requires an infinite number of points. To partially compensate for the finite data set a modified data set is created by multiplying the experimental data set by an arbitrary window function. This process does not completely compensate for the finite data set and a number of arbitrary window functions can be used that provide various types of partial compensation. See pages 423-428 of reference 2. The window functions that are included with this software are defined in the following ways. Where j is the data point number and n is the total number of data points: -

$$\text{Parzen} = 1-\left|\frac{(j-0.5(n-1))}{(0.5(n+1))}\right| \qquad (2.1)$$

$$\text{Hanning} = 0.5\left(1-\cos\left(\frac{2\pi j}{n}\right)\right) \qquad (2.2)$$

$$\text{Welch} = 1-\left(\frac{(j-0.5(n-1))}{(0.5(n+1))}\right)^2 \qquad (2.3)$$

$$\text{Square} = 1 \qquad (2.4)$$

The FFT spectrum of a chaotic system has two parts – a continuous but noisy background and a number of dominant peaks which are difficult to observe because of the noise. However, the dominant peaks may be readily observed by averaging a number of spectra for the same pendulum parameters. In addition the program can enhance spectral characteristics by averaging a number of spectra. The software has this capability. At the start you will be asked for the file name of the phase data and then the maximum number of spectra that may be averaged is shown. The number of spectra that you choose to average must be less than or equal to the maximum number shown. You should save at least 37000 data points for each FFT calculation. The total number of points to save is 37000 times the number to be averaged.

Another measure that differentiates chaotic and non-chaotic motion is the Lyapunov exponent and this is calculated after the FFT is completed. A trajectory is defined by the variables $\left(\theta, \frac{d\theta}{dt}\right)$ as a function of time. A trajectory will change when the initial conditions are changed from $\left(\theta_1(0)\left(\frac{d\theta_1(0)}{dt}\right)\right)$ to $\left(\theta_2(0)\left(\frac{d\theta_2(0)}{dt}\right)\right)$. Where $\theta_1(0)-\theta_2(0)$ is small. In a non-chaotic mode the trajectory will coalesce to a single trajectory even though the initial conditions are different. However, in a chaotic mode a new trajectory will diverge away from a previous trajectory. As long as the two initial conditions differ by only a small amount, the difference

$\varepsilon(t)=\sqrt{(\theta_1-\theta_2)^2+\left(\frac{d\theta_1}{dt}-\frac{d\theta_2}{dt}\right)^2}$ will follow an exponential divergence - $\varepsilon(t)=\varepsilon(0)^{\lambda' t}$ as long as t is small. The complete data set is divided into n short data sets and the local exponent, λ', for each short data set is calculated. In a large data set the local exponent, λ', will vary. The Lyapunov exponent, _, is the average of many local exponents throughout the data set.

$$\lambda=\frac{1}{n}\sum_{i=1}^{n}\mathrm{Ln}\left(\frac{\varepsilon_i(t)}{\varepsilon_i(0)}\right)$$ Where n is large.

The Lyapunov exponent is calculated and displayed after the FFT spectrum has been displayed. An algorithm that was developed by Wolf et al (See ref. 9) has been used.

T	Print	Dim

Print causes any Phase, Poincaré or Power spectra plots that are displayed on the screen, to be sent to the printer.

rint	Dimension	Simul.

Dimension is a measure of the complexity of a Poincaré plot. There are many definitions of dimension but capacity dimension is one that can readily be applied to experimental data and it is calculated here. A Poincaré plot that comprised a single point would have dimension 0. Whereas a Poincaré plot that had every site on the plot filled with a point would have a dimension 2. A continuous line would have dimension 1. In practice all chaotic Poincaré plots will have a dimension that is somewhere between 1 and 2.

Imagine that a one-dimensional structure (a line) is of length, L, and that it is covered by boxes of length, ε. There are $N(\varepsilon)=(L/\varepsilon)$ boxes.

Now imagine that a structure that is a square of side length, L. The square is covered by boxes of side length, ε. There are $N(\varepsilon) = (L/\varepsilon)^2$ boxes. Considering a three dimensional object $N(\varepsilon) = (L/\varepsilon)^3$. In general, for an object of dimension D the number of boxes is $N(\varepsilon) = (L/\varepsilon)^D$. The logarithm of N(_) is given by

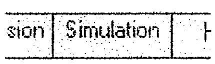
$$\ln(N(\varepsilon)) = D\ln(L) + \ln(1/\varepsilon) \qquad (2.5)$$

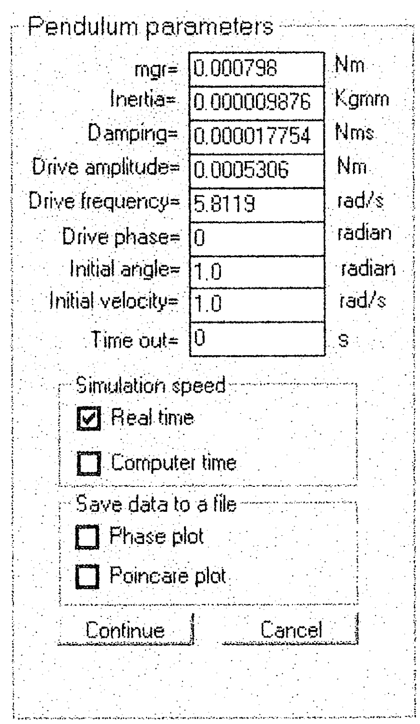
That is, $$D = \frac{\ln(N(\varepsilon))}{\ln(L) + \ln(1/\varepsilon)} \qquad (2.6)$$

If $\varepsilon << L$ then $\ln(L) << \ln(1/\varepsilon)$ (2.7)

$$D = \frac{\ln(N(\varepsilon))}{\ln(1/\varepsilon)} \qquad (2.8)$$

In practice, a Poincaré plot is covered by boxes of side, ε, and N(ε) is found by counting the number of boxes that have at least one point in them. This is repeated for a range of values of ε. A straight line is fitted to a plot of $\ln(N(\varepsilon))$ versus $\ln(1/\varepsilon)$ and the slope is the capacity dimension, D. When the Dimension button is activated it will first ask for a file of Poincaré data which will be plotted to the screen and then the capacity dimension and the number of points will be displayed.





Simulation is a numerical routine that uses a fourth order Runge Kutta integration algorithm to solve the driven pendulum equation and display the data in the same way that the experimental data is shown on the screen. When the simulation button is activated the upper left logo switches from the experimental pendulum to a laptop computer and screen and a data panel (shown below) immediately appears. The equation parameters may be changed in this data panel. The default values approximate to those of an experimental pendulum. But do bear in mind that no two pendulums are exactly alike. Eq. 2.9 is the equation of motion of a damped driven pendulum.

$$I\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + mgr\sin(\theta) = A\cos(\Omega t + \Phi) \qquad (2.9)$$

where

mgr is the product of the weight and radius of gyration of the bob structure,

I is the moment of inertia,

b is the damping constant,

A is the amplitude of the drive torque,

Ω is the drive frequency, and

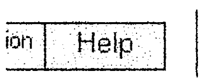
Φ is the phase of the drive torque. The Poincaré data is recorded when modulo Ω t = 2π.

The default parameter values (I, b, A, Ω and mgr) are shown in the screen panel. The parameters (I and mgr) were set to be as close as possible to those of one experimental pendulum. Other experimental pendulums may have slightly different values. The default values of (b, A, Φ, Θ) were chosen to set the pendulum in a chaotic state.

The speed of simulation may be set to real time by clicking on the "Real time" box in the simulation speed frame or to computer time by clicking on the Computer time box. Computer time will be much faster.

Clicking on either of the Phase plot box or the Poincaré plot box in the "Save data to a file" frame followed by the continue button will cause a request for the number of data points panel followed by request for a file name panel. The simulated data will be saved to a file at the same time as it is generated.

Time out allows you to set a time period where data is calculated but it is not displayed or recorded. This is a useful feature for identifying periodic modes that may have a settling time.



Help activates the help file.

3. Calibration Procedures

Equation of motion of a driven pendulum

The pendulum may be described as a mass m suspended a distance r from the system axis. The total moment of inertia, I, is due to both this mass and the inertia of all the additional rotating components attached to the axle (primarily the optical encoding wheel and the annular ring magnet). If a torque T is applied to the system via the motor then the equation of motion of the pendulum is

$$I\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + mgr\sin(\theta) = T \qquad (3.1)$$

where b is the damping coefficient arising from the electrodynamic interaction between the rotating ring magnet and the fixed eddy-current plate. T is the torque that is applied by the torque converter.

(NOTE. In the following sections, actual data obtained from a pendulum will be presented. It should be noted that the precise values of system parameters extracted from the associated graphs are specific to a particular apparatus. Somewhat different numbers will be obtained for a different pendulum/drive-module combination. In other words, these results are included only in order to suggest the general appearance of results, which may be expected, and as an aid in carrying out the calibration process.)

Experiment #1

To determine the natural frequency of the pendulum

Release the lighted pushbutton (on the front panel) to the OFF position, thereby preventing any drive circuit leakage current from flowing through the motor coils; this will eliminate any residual torque offset. Adjust the micrometer so that the copper plate is well separated from the ring magnet (1 cm or more); b is thus approximately zero. Turn off any torque signal (T=0). The pendulum equation in this case becomes

$$I\frac{d^2\theta}{dt^2} + mgr\sin(\theta) = 0 \qquad (3.2)$$

For small angles, $\sin(\theta) \approx \theta$ and the oscillation frequency is

$$\omega_0 = \sqrt{\frac{mgr}{I}} \qquad (3.3)$$

The period of oscillation for arbitrary maximum angular displacement _$_m$ can be found analytically from the differential equation, and is

$$P = P_0\left[\frac{2}{\pi}K(k)\right] \qquad (3.4)$$

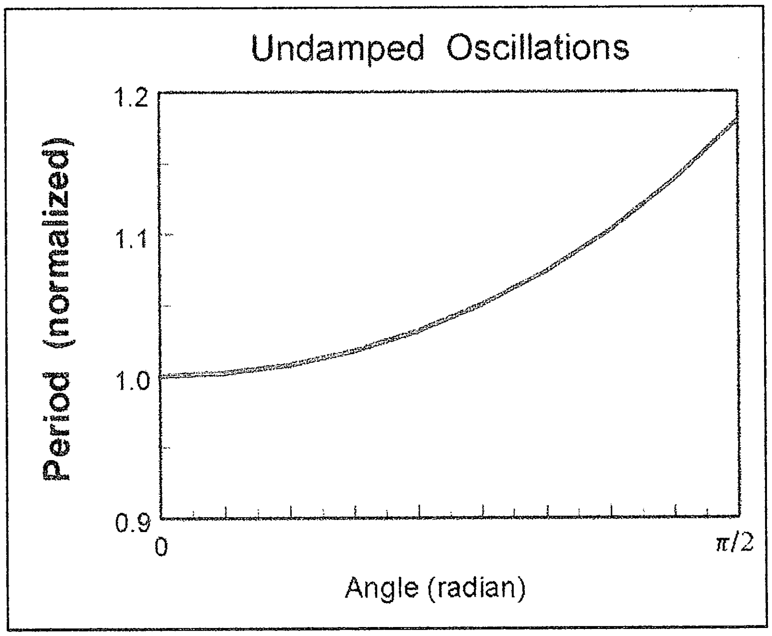
where K(k) is a complete elliptic integral of the first kind, k=sin(_$_m$/2), and P_0 is the pendulum period in the limit of infinitesimally small oscillations. The variation of period with maximum angular displacement is illustrated in figure3.1 where the normalized period is P/P_0.

Experimental procedure

To determine P_0 proceed as follows. Displace the pendulum manually through and angle _$_m$ and release it. Using the Save orbit button record a time series of n data points (n should be chosen large enough to include at least five oscillation cycles; i.e. approximately 3000 points. An analysis of this data (by means of a computer program or visually inspecting the data) for zero crossings will give the pendulum period P for the oscillation amplitude θ_m. Equation 3.4 will then yield P_0.

An alternate procedure would be to measure the pendulum period for a series of diminishing maximum angles and then extrapolate the data to the zero-amplitude period P_0, as indicated in the figure. By either method the natural frequency is then evaluated from $\omega_0 = 2\pi / P_0$.

Figure 3.1:



Experiment #2

To determine the damping b/I as a function of the micrometer reading

Method 1

Turn off any torque signal, release the lighted push button to disconnect the drive circuit, and set the micrometer to a desired value. Manually displace the pendulum bob through a small initial angle, release it and record the resulting motion in the form of a time series $\theta(t)$. Under these conditions, the general equation of motion reduces to

$$I\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + mgr\,\theta = 0 \qquad (3.5)$$

In the underdamped case (_>b/2I), the analytic solution is

$$\theta = \theta_0\, e^{-\alpha t} \cos(\omega_1 t) \qquad (3.6)$$

where _ = (b/2I) and $\omega_1^2 = \left(\omega_0^2 - \alpha^2\right)$.

Time has been normalized in units of P_0. Since ω_0 has already been determined from Experiment #1, and θ_0 is just the initial angle. The positive or negative peaks of the time series for θ can be least squares fit to a semilog plot of $e^{-\alpha t}$ and therefore α, for a particular micrometer setting, can be determined from the slope of the resulting straight line. The relative damping parameter is then given by $b/I = 2\alpha$ By repeating this procedure for a number of micrometer settings, a calibration curve of b/I versus damping plate separation can be constructed.

Method 2

Turn off any torque signal, release the lighted pushbutton to disconnect the drive circuit, and set the micrometer to a desired value. Turn the entire apparatus on-end so that the pendulum mass will now move in a horizontal plane (be careful to support the apparatus only on its rigid frame, and not on either the end bearing or the sliding top plate). This orientation effectively turns gravity OFF, in which case the governing differential equation becomes.

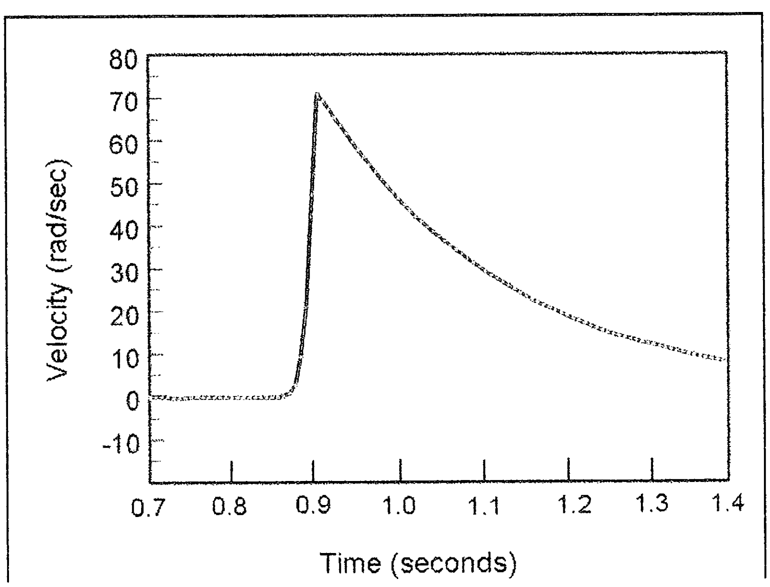
$$I\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} = 0 \qquad (3.7)$$

If the pendulum is given an initial spin by hand, then its angular velocity will simply decay with time according to the expression

$$\frac{d\theta}{dt} = \left[\frac{d\theta}{dt}\right]_0 \exp[-(b/I)t] \qquad (3.8)$$

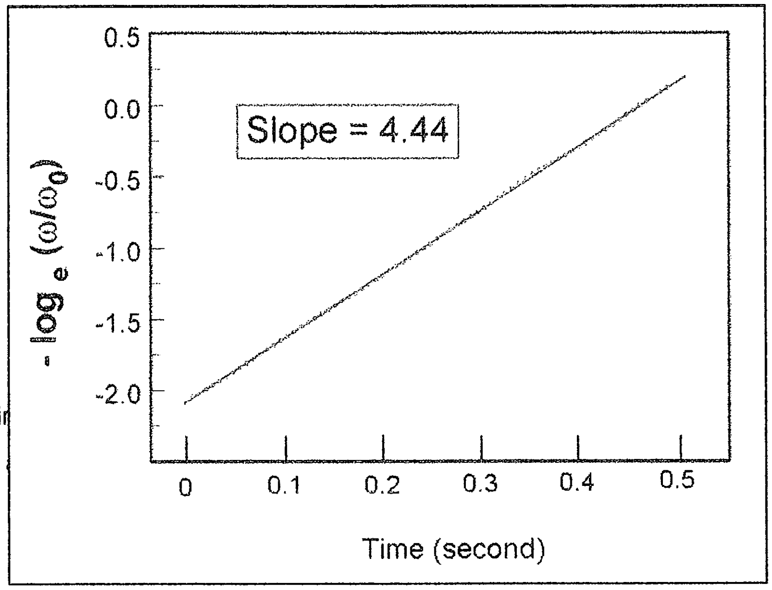
Experimental data are displayed in figure 3.2. The procedure used was to choose a micrometer setting, select the data acquisition parameters (file name, number of points, etc.), initiate the data collection, then immediately set the system spinning with a careful flick of the finger. Several hundred data points were sufficient. The initial zero velocity segment of the plot is just that portion of the data prior to the impulse that set it in motion. Recall that angles and velocities are sampled at about 2 millisecond intervals.

Figure 3.2:



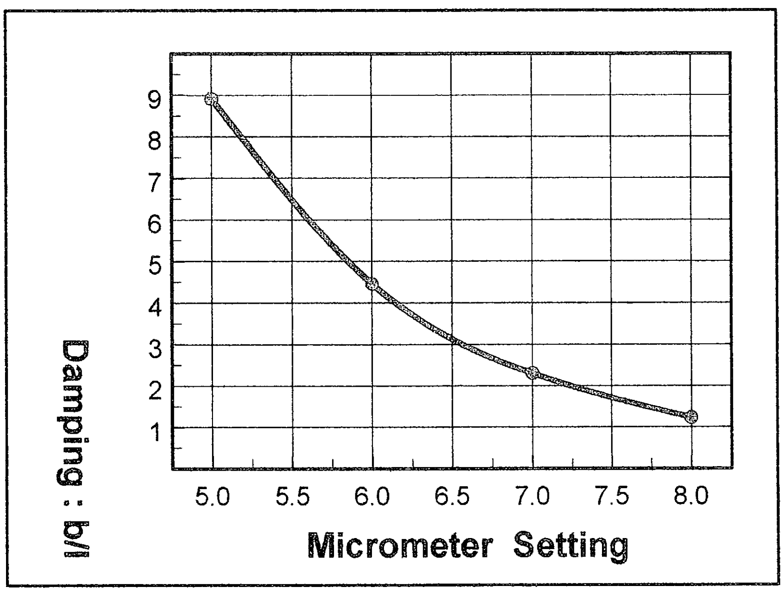
The decay portion of the data in fig. 3.2 can be replotted in the form of the natural log of the normalized angular velocity versus time; this is shown in fig. 3.3.

Figure 3.3



The linearity of this graph confi
method 1) can be repeated for
shown in Fig.3.4.

Figure 3.4



Experiment #3

To determine the relationship between input voltage and torque

IMPORTANT: disable the internal voltage by releasing the lighted pushbutton to its OFF state.

Turn the apparatus on-end so that gravity is OFF, and begin by applying about 1 volt from a separate DC supply to the "Torque Calibrate" inputs at the rear of the drive unit. Adjust the micrometer (damping) so that the pendulum is spinning at no more than 1000 rpm. Under these conditions, the equation of motion is

$$I\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} = T \qquad (3.9)$$

The pendulum will accelerate until a terminal velocity is reached. Setting $d^2\theta / dt^2 = 0$ in Equation (3.9) yields

$$T = b\left[\frac{d\theta}{dt}\right]_{term} \qquad (3.10)$$

The critical torque of the pendulum is T_c = mgr; this amount of applied torque would cause the pendulum to be displaced by 90 degrees from the vertical. An infinitesimal increase in T above T_c would set the pendulum into rotation. In normalized units T_c/I = mgr/I. But the natural frequency of the pendulum is given by $\omega_0^2 = mgr / I$, and so $T_c / I = \omega_0^2$ where ω_0 is known from Experiment #1. So

$$\frac{T}{T_c} = \left(\frac{1}{\omega_0^2}\right) \left(\frac{b}{I}\right) \left[\frac{d^2\theta}{dt^2}\right]_{term} \qquad (3.11)$$

and the first two bracketed terms on the right hand side are already known from previous experiments. By measuring the terminal velocity for several different applied voltages, it is possible to construct a plot of torque T / T_c versus V as shown in Fig. 3.5. Such a plot will reveal deviations from linearity in the drive circuit; these typically will appear at larger applied voltages.

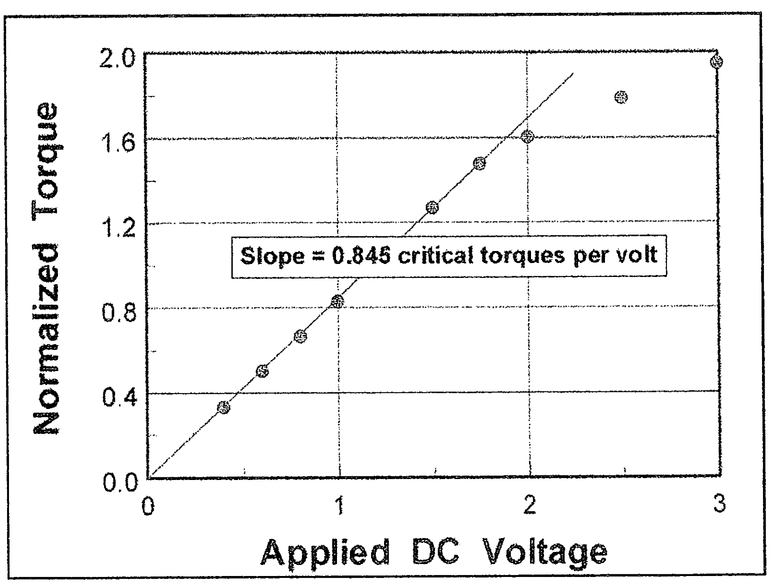


Figure 3.5

4. Resonance

If the pendulum is subjected to a harmonic applied torque, which is sufficiently weak that the resulting oscillations are of small amplitude (and hence $\sin(\theta) \approx \theta$), then the governing equation becomes

$$I\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + mgr\ \theta = T_0 \sin(\omega t) \tag{4.1}$$

The steady state solution of this equation is given by

$$\theta = \frac{T_0 / I}{\sqrt{(\omega_0^2 - \omega^2)^2 + (b/I)^2\omega^2}} \sin(\omega t - \phi) \tag{4.2}$$

where

$$\tan(\phi) = \frac{(b/I)\omega}{\omega_0^2 - \omega^2} \tag{4.3}$$

After the initial transients have died away, the pendulum will oscillate at the drive frequency _, but with a phase shift ϕ. Figure 4.1 illustrates the amplitude response of the system for $\omega_0 = 1.0$ and $T_0 / I = 0.25$. For a given damping b/I, the oscillations will have maximum amplitude at a drive frequency somewhat less than the natural frequency of the pendulum. In fact the expression for this optimum drive frequency is

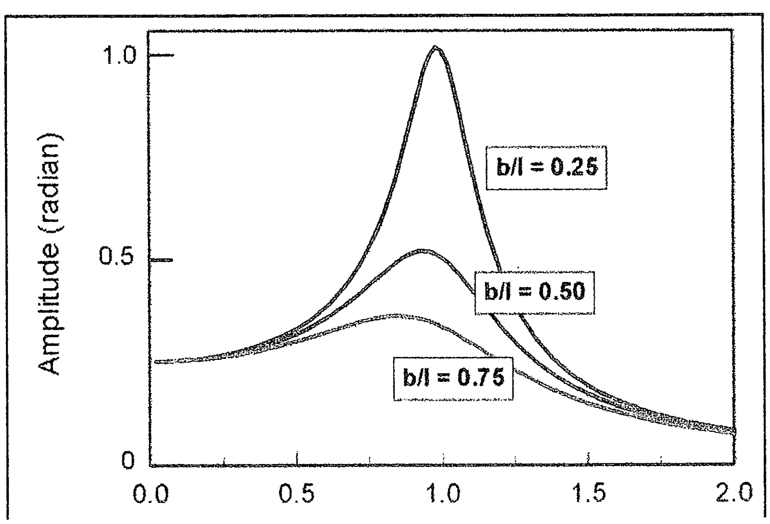
$$\omega^2 = \omega_0^2 - (b/I)^2/2. \tag{4.4}$$

Equation (4.1) may be expressed in the form

$$\frac{d^2\theta}{dt^2} + \frac{\omega_0}{Q}\frac{d\theta}{dt} + \omega_0^2\,\theta = \frac{T_0}{I}\sin(\omega t) \tag{4.5}$$

where $Q = \dfrac{\omega_0}{b/I}$

Figure 4.1:

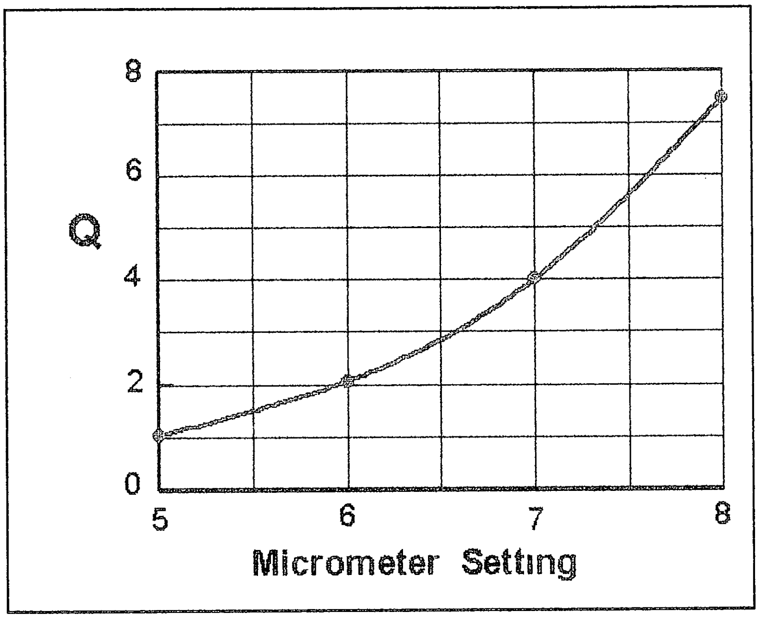


Experiment #4

To study the resonant response of the pendulum

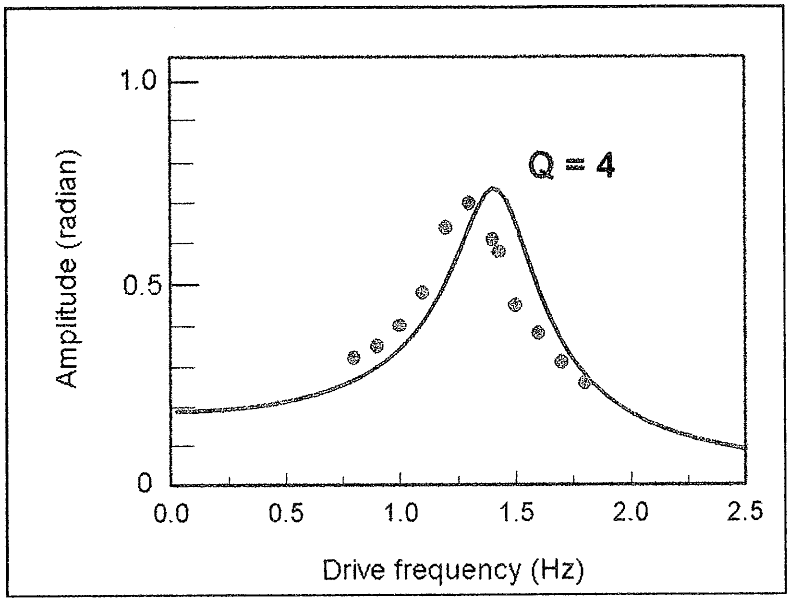
Adjust the micrometer so that the Q is about 4 or 5 (this is determined by b/I and already determined ω_0). Set the amplitude of the internal sine-wave generator to a value that produces pendulum oscillations of about 20 to 30 degrees when the drive frequency equals $\omega_0 / 2\pi$. Now select a number of frequencies for the sine-wave generator, ranging from about $0.5(\omega_0 / 2\pi)$ to $1.5(\omega_0 / 2\pi)$, and for each frequency determine the average of the maximum displacements of the pendulum to the right and the left. This can be done

Figure 4.2:



by recording and then examining a time series $\theta(t)$ for each drive frequency. A plot of θ_{max} versus ω (or f) will exhibit the resonance peak. Typical experimental data are included in Fig.4.3. This experiment can be repeated for several different values of Q. Compare your resonance curves with the predictions of Equation (4.2) as represented by the solid line in Fig. 4.3. Repeat the procedures above, but with drive amplitude for which the oscillations are much larger amplitude – say about 60 or 80 degrees. Discuss any deviations, which are now observed between the experimental data and Equation (4.2).

Figure 4.3:


Q = 4
Amplitude (radian)
1.0
0.5
0
0.0
0.5
1.0
1.5
2.0
2.5
Drive frequency (Hz)

5. Hysteresis

Hysteresis is well known in the B – H behavior of a ferro-magnet and in the force – acceleration plot of an object on a rough horizontal surface. Hysteresis may also be observed in the torque – angular velocity plot of a driven pendulum. If an applied dc torque is very slowly increased from zero, the pendulum angle will equally slowly increase. When the torque reaches the critical value T_c, the angle will be 90 degrees with respect to the vertical. A slight increase in T above T_c will cause the pendulum to suddenly begin to rotate. The angular velocity will not be constant, but instead will undulate slightly between a maximum value (pendulum is at its lowest point) and a minimum value (pendulum is at its highest point). Let the time average of $d\theta / dt$ be denoted $\langle d\theta / dt \rangle$. A hypothetical plot of T / T_c versus $\langle d\theta / dt \rangle$ is shown in Fig.5.1. Notice that once the pendulum is rotating, the torque may be decreased below the value to T_{min} before an abrupt switch to an oscillating but non-rotating state with $\langle d\theta / dt \rangle = 0$ will take place. This is an example of hysteresis, which, for the pendulum, is due to inertia.

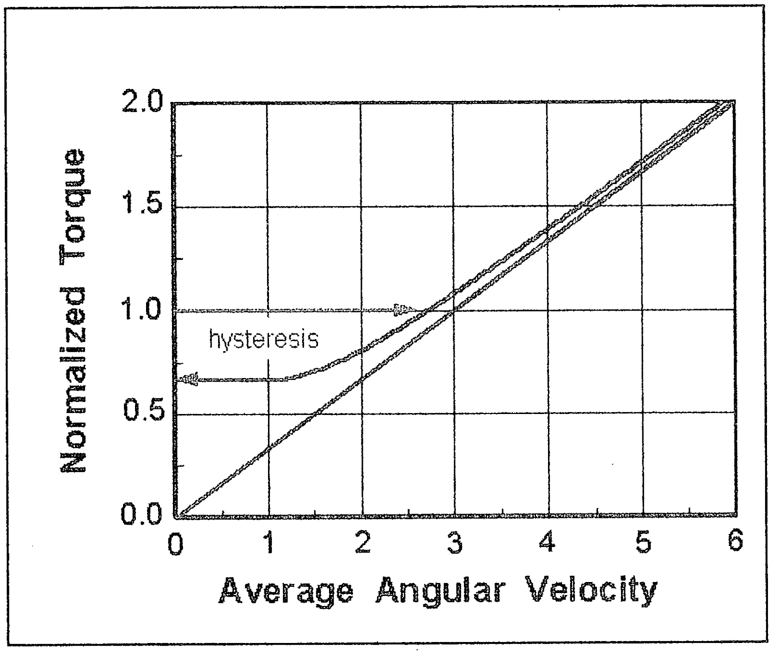
Experiment #5

<u>To observe hysteresis in the driven pendulum</u>

IMPORTANT: disable the internal voltage source by releasing the lighted pushbutton to its OFF state.

Set the damping for a Q of about 4. Apply torque voltages from an external DC source in a series of steps beginning just below V_c and increase to about $2V_c$. At each voltage, record a time series $\theta(t)$ and $d\theta / dt$. From this sequence obtain the average $\langle d\theta / dt \rangle$, and plot these points horizontally versus normalized torque vertically. Now decrease the torque voltage in steps, obtaining $\langle d\theta / dt \rangle$ for each V, until a transition occurs to a non-rotating state. Plot these points and note the amount of hysteresis. Carry out the whole procedure for a different Q value and note the change of hysteresis.

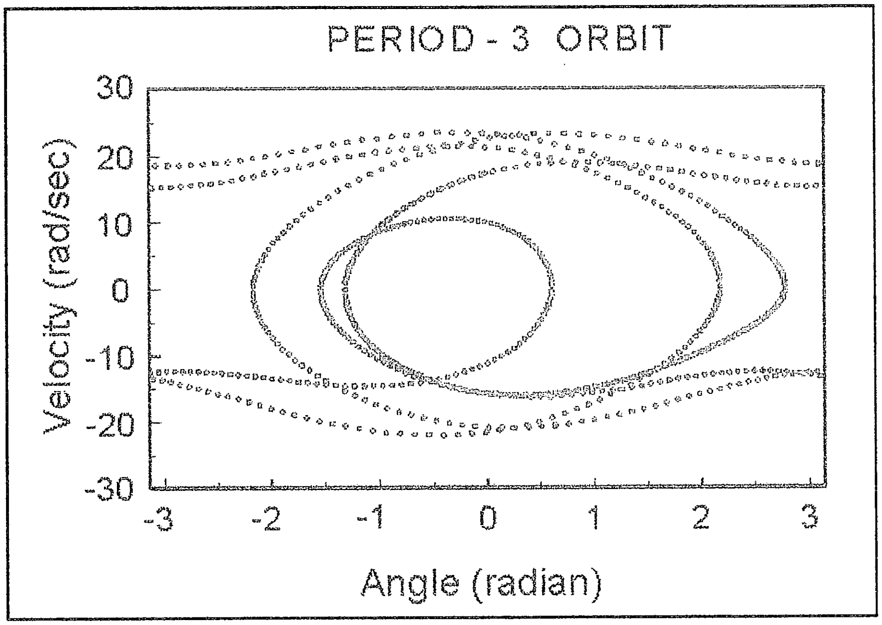
Figure 5.1:


2.0
1.5
1.0
0.5
0.0
Normalized Torque
hysteresis
0
1
2
3
4
5
6
Average Angular Velocity

6. Multiperiodic and chaotic behavior

As noted in section 1 (Introduction), two special plots are usually constructed from the pendulum's observed time series $\theta(t)$ and $d\theta/dt$.

Figure 6.1:



A phase plane orbit is a representation of the motion formed by plotting the succession of coordinate pairs $(\theta, d\theta/dt)$. If the displacement θ were measured relative to the vertically downward rest position, the simple oscillating motion would, for example, appear as an elliptical orbit in the phase plane and would be contained within the range $-\pi < \theta < \pi$. When rotations (flipping over the top) occur, two strategies can be followed:

Figure 6.2:

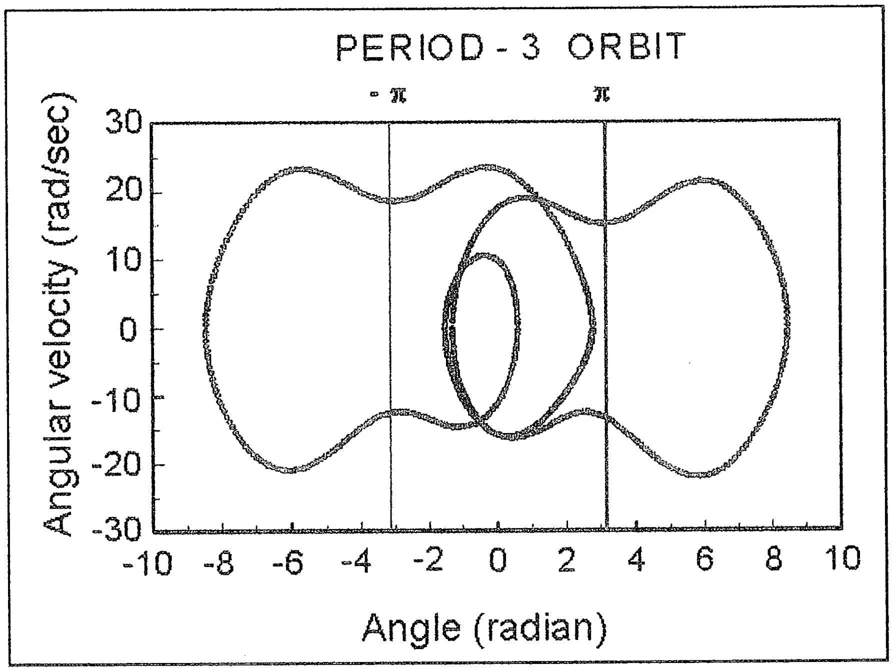
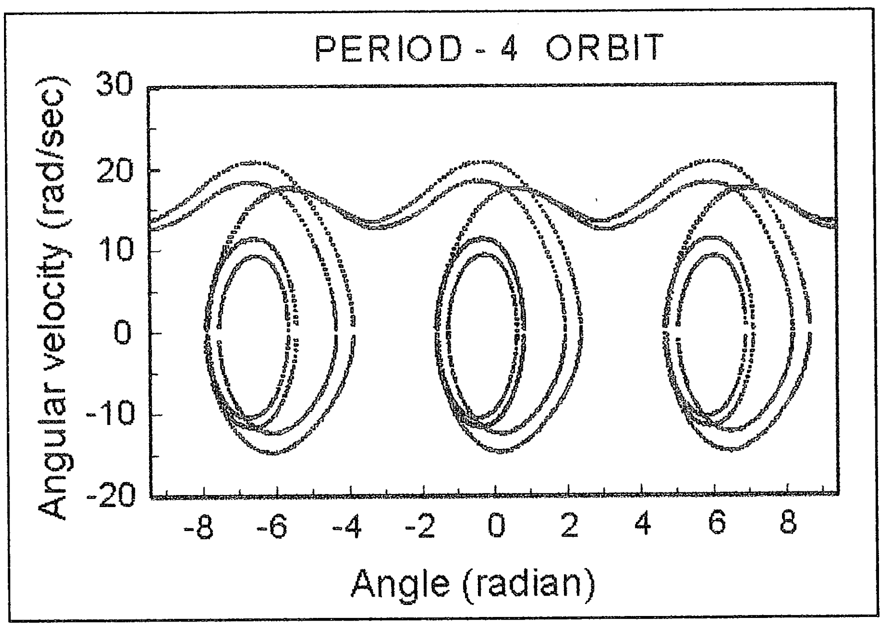


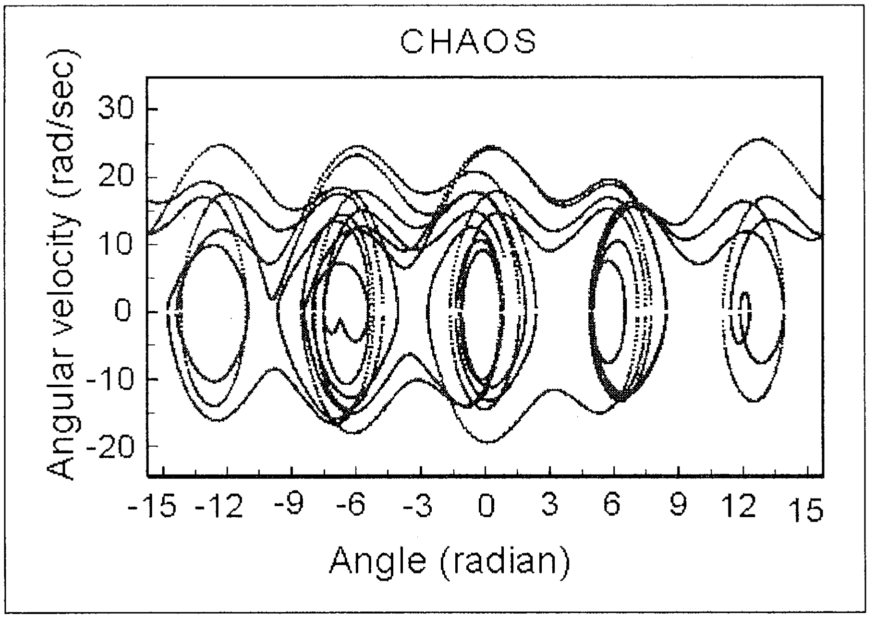
Figure 6.3:



(a) the range of θ can be extended to $(-3\pi, 3\pi)$, $(-5\pi, 5\pi)$, etc. (b) motion outside of the principal domain $(-\pi, \pi)$ can be folded back into this domain. This second choice is the one very commonly followed in the literature. Its advantage is compactness; its disadvantage is that the folding can obscure certain features in the orbits. As an example, consider Figures 6.2 and 6.3 that are plotted from the same experimental set.

In this particular situation, the unfolded representation conveys the repetiveness of the closed phase plane orbit. In cases where an overall winding (clockwise or counterclockwise) is superimposed on the oscillations, it will not be possible to contain the orbit within any finite range of _. Nevertheless, an extended range can still help in visualizing the motion, as can be seen in Fig.6.4.

Figure 6.4:



We also introduced the Poincaré plot in section 1. It is obtained by sampling the pendulum motion once only in each period of the sinusoidal drive torque and it can be thought of as a stroboscopic portrait of the phase plane motion. Period 2 motion of the pendulum would produce a Poincaré plot containing just two points. And so forth. Because these maps represent stable periodic states of the system, they are called attractors.

In section 4 on resonance, a weak harmonic drive torque was applied to the pendulum at frequencies bracketing the natural frequency ω_0. It was seen that the resulting steady state motion was also harmonic at ω, phase shifted by ϕ, and had maximum amplitude when $\omega^2 = \omega_0^2 - (b/I)^2/2$. More complex motion of the pendulum may result from the application of a stronger sinusoidal torque to this nonlinear system at lower frequencies. Because the differential equation for the pendulum

$$I\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + mgr\sin(\theta) = T_0\sin(\omega t) \qquad (6.1)$$

is deterministic, the "classical" expectation would be that periodic or multiperiodic motion would arise. In other words, the oscillations would remain repetitive, but with an overall cycle time of perhaps several drive periods. While this picture is indeed correct at relatively low drive amplitudes (for which non-linearity is weakly contributing) or large drive frequencies, that is, T > T_c and _< $_0$. Within this domain of (T, _), chaos can appear. More information on this extensive subject can be found in reference 11.

For the pendulum, chaos is manifested as oscillations *and/or* rotations, which contain no repetitive pattern. The orbit in phase space $(\theta, d\theta/dt)$ is endlessly changing. The motion is, therefore intrinsically unpredictable.

The chaotic dynamics of the driven pendulum can be explored with this apparatus by choosing various drive amplitudes and frequencies within the domain mentioned previously and then observing both phase plane orbits and Poincaré maps. A Q between 4 and 5 is suitable.

Figure 6.5:

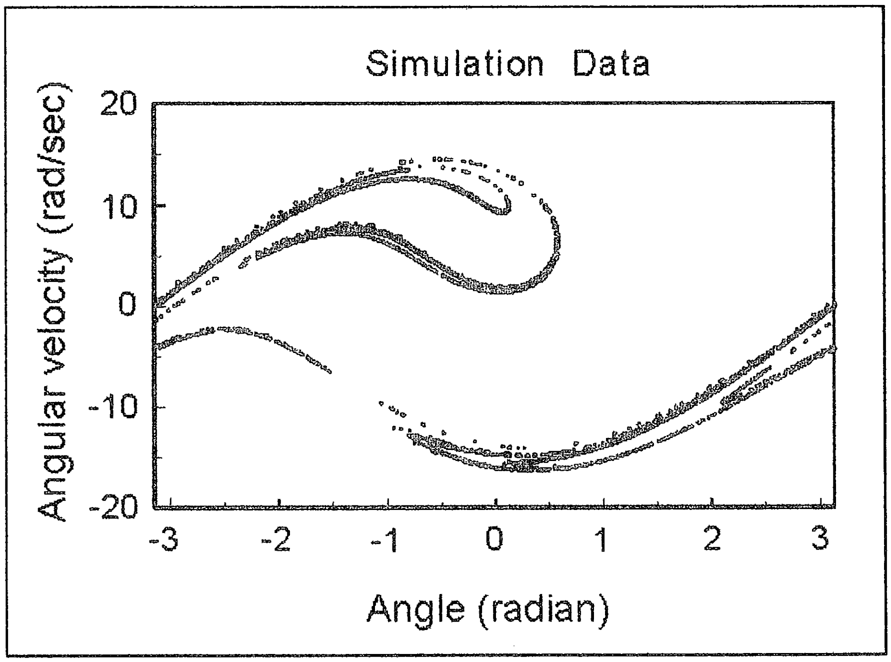


Figure 6.6:

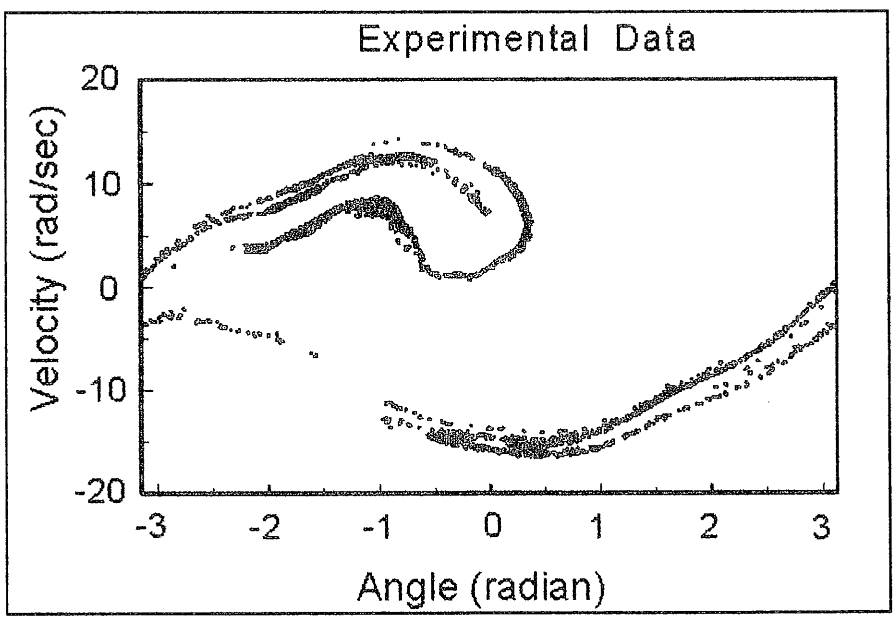
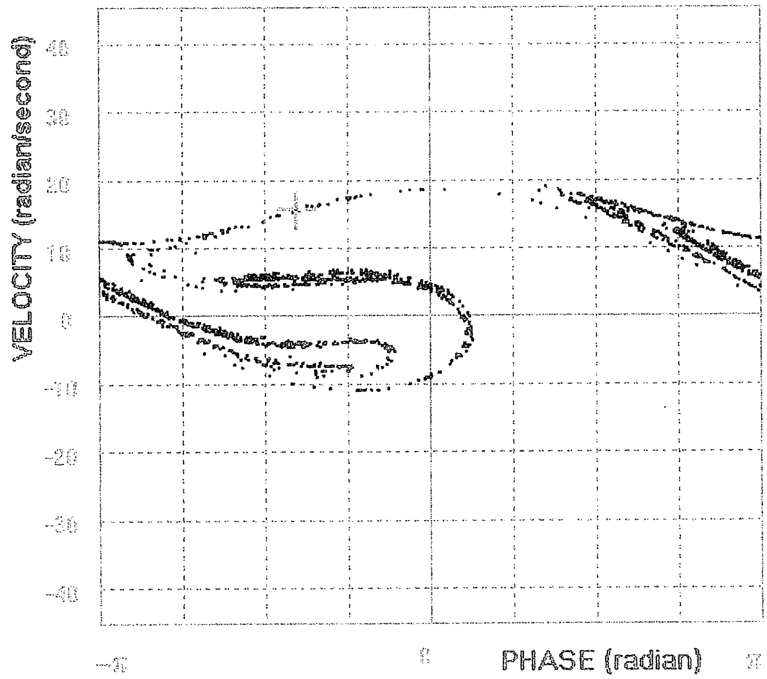


Figure 6.7 is a Poincaré plot from the screen that was generated by the simulation using the default parameters. The Lyapunov exponent (0.312) and the dimension (1.1878) both give a measure of the degree of chaos whereas an FFT spectra can give some detail of the structure of the chaos in frequency space.

Figure 6.7



Poincare plot

Using the same parameters a time series of 1000000 points of the velocity was recorded. Since the angle coordinate has discontinuities, the velocity was chosen for the FFT calculation. The FFT is shown in fig. 6.8. The vertical decibel scale is calculated from $10\log_{10}$(relative power).

Figure 6.8

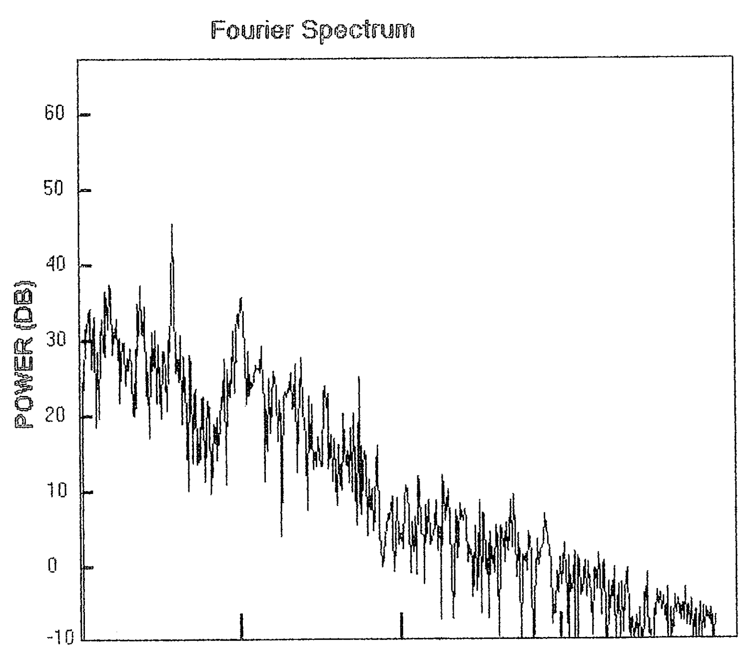


Figure 6.8 shows a rather noisy spectrum that decreases approximately as 1/f. Figure 6.9 shows the average of 5 FFT's.

Figure 6.9

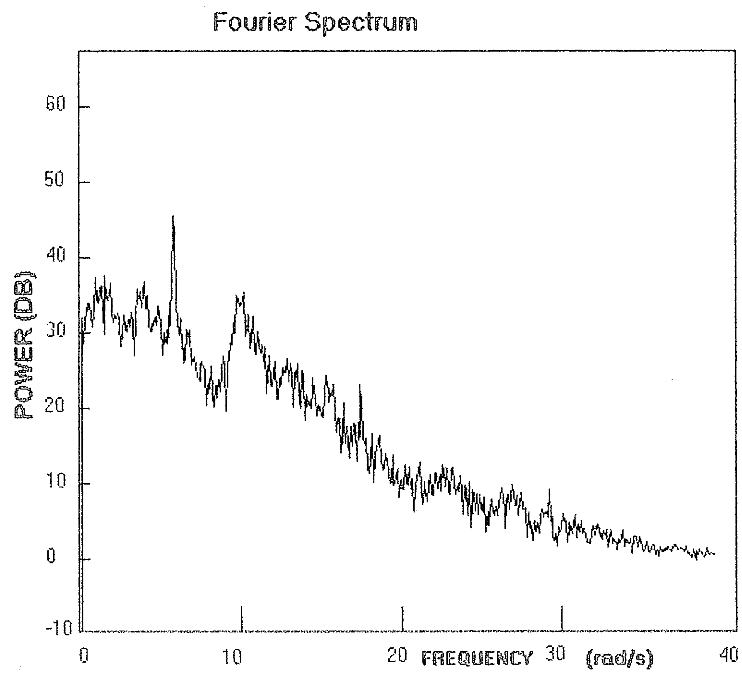
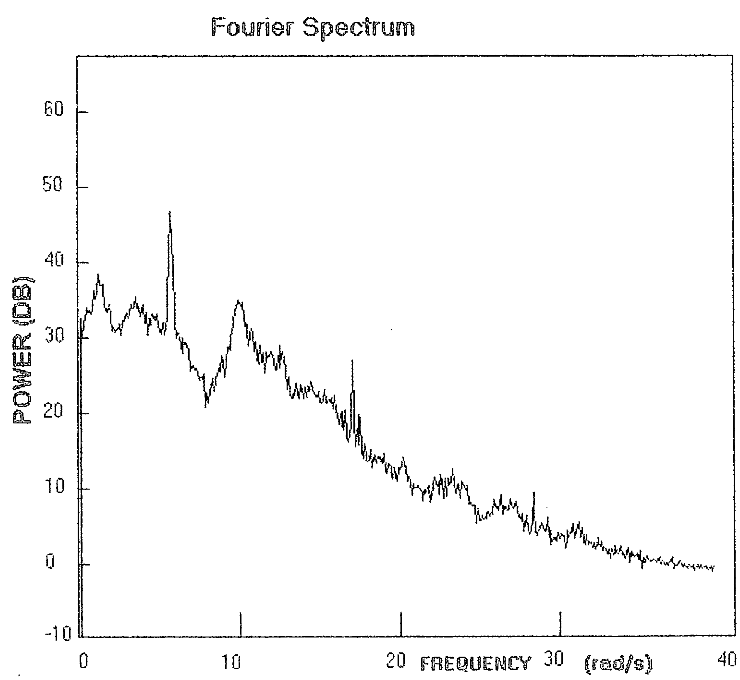


Figure 6.9 shows less noisy continuous spectrum with a sharp peak at the drive frequency 5.8 r/s and a broad peak centered at about 10 r/s. The low amplitude natural resonance frequency is at 8.99 r/s. There is a hint of higher harmonics but it would not be prudent to make such a conclusion from this plot.

Figure 6.10 is the average of 20 FFT's.

Figure 6.10



The drive frequency is the peak visible at 5.8 r/s with the third harmonic at about 17 r/s and the fifth harmonic at 29 r/s. (Odd harmonics are characteristic of a periodic nonlinear pendulum. (Ref. 10)) A number of other peaks are becoming clearer including that at about 10 r/s. During chaotic motion there will be short time periods when the pendulum is synchronized to the drive frequency which gives rise to the sharp peaks in the FFT spectra. At other short bursts of time the pendulum will be oscillating at its resonant frequency. Nonlinearity of the pendulum makes the resonant frequency a function of the amplitude, giving rise to breadth in the resonant peaks. Clearly the FFT can yield information that neither the dimension nor the Lyapunov exponent has.

7. Suggested readings

1. *The Pendulum – a case study in physics,* Gregory L. Baker and James A.Blackburn, Oxford University Press (2005). A comprehensive discussion of linearized and nonlinear periodic pendulums, as well chaotic and coupled pendulums.
2. *Numerical Recipes,* W.H. Press, S.A. Teukolsky, B.P. Flannery, and W.T. Vetterling, Cambridge University Press (1986). A useful algorithm for computing complete elliptic integrals is given on pages 187-188. See pages 423-428 for a discussion of windows that can be used with FFT.
3. *Mathematical Methods in the Physical Sciences*, Mary L. Boas, Wiley (1966). See pages 346-348 for a discussion of resonance in harmonically forced linear oscillator.
4. *Microelectronic Circuits*, Second Edition, A.S. Sedra and K.C. Smith, Holt, Rinehart and Winston (1987). See pages 718-719 for resonance plots for a second order system.
5. *Modern Control Systems*, 2nd Edition, R.C. Dorf, Addison-Wesley (1974). See page 207 for resonance plots for a second order system.
6. *Chaotic Dynamics: an introduction*, Second Edition, G.L. Baker and J.P. Gollub, Cambridge University Press (1996). A compact introduction to chaos, and nonlinear dynamics with extensive reference to the driven damped pendulum. See, in particular, pages 135-139.
7. *Chaotic Vibrations*, F.C. Moon, Wiley (1987). An excellent discussion of a variety of chaotic systems and a readable account of theoretical techniques for analyzing chaotic behavior.
8. *Chaotic Dynamics of Nonlinear Systems*, S.N. Rasband, Wiley (1990). An accessible introduction to the theory, techniques, and applications of chaos. Slightly deeper and more formal than Moon's book.
9. *Determining Lyapunov exponents from experimental data,* A. Wolf, J.B. Swift, H.L. Swinney and J.A. Vastano, Physica **16D,** 285-317 (1985).
10. *Probability, pendulums, and pedagogy,* Gregory L. Baker, **74**, Am. J. Phys. 482-489 (2006) A discussion of some characteristics (including frequency spectra) of the increasingly complex behavior of the pendulum as developed from the linearized approximation to the fully chaotic state.
11. *Experimental study of Chaos in a damped Driven Pendulum,* J.A. Blackburn, Yang Zhou-ing, S. Vic, H.J.T. Smith and M.A.H. Nerenberg, Physica **26D**, 385-395 (1987).

8. Suggested exercises

1. **The FFT of a chaotic time series**

Adjust the pendulum parameters until it is in a chaotic mode. Record a long phase plot time series. Plot the FFT. Note that there are a number of peaks and a noisy but continuous background. Measure the frequencies of the peaks. Are the frequencies of the peaks multiples of either the drive frequency or the small angle resonant frequency or are they multiples of the difference of the drive and resonant frequency. Can the continuous background of the FFT be fitted to a 1/f function?

2. **Capacity dimension of a Poincaré plot**

The aim of this exercise is to investigate how the estimate of the dimension varies with the number of data points. Adjust the pendulum parameters until it is in a chaotic mode. Record sets of Poincaré data to files with 100, 200, 500, 1000, 2000 points. Find the capacity dimension for each data set. Make a plot of dimension versus number of data points. Note that the dimension increases with the number of points up to a plateau region. Estimate the least number of points at the plateau region for a consistent estimate of dimension.

3. **Simulation**

The aim of this exercise is to reconcile experimental data with simulation data. Adjust the pendulum parameters until it is in a periodic mode. The phase plane plot should look like an ellipse. Note the pendulum parameters, Q (or *b/I*) and the drive frequency and amplitude of the drive. Activate the simulation button and set these parameters into the panel that has appeared. Run the simulation. The simulated phase plot should look like the experimental phase plot. If it does not, make small adjustments to the simulation parameters until the phase plots look the same. Repeat with the pendulum in a multi-periodic mode. You will find that it is more difficult to obtain agreement between experimental and simulated plots in the multi-periodic mode. Now set the pendulum into a chaotic mode. The phase plots will be too complex too yield any information. Record an experimental Poincaré plot. Set the simulation with the same parameters as the experimental pendulum. If the experimental and simulated Poincaré plots are not similar adjust the simulation parameters in small increments until they do agree. This may take some time, as the chaotic mode is very sensitive to parameter values.

4. **FFT window functions**

The purpose of this exercise is to observe the effect of the various FFT window functions on an FFT spectra. Adjust the pendulum parameters until the experimental pendulum is in a chaotic mode. Record a phase plane set of data (~50000 points). Click on the FFT button and select a window function. Plot the FFT and print it. Repeat for each of the window functions. Note the differences among the four spectra.

5. **Temporary chaotic behavior**

On setting a system into motion, it is possible for it to show chaotic behavior initially before settling down to periodic or multi-periodic behavior. Click on the simulation button and change the drive frequency to 5.0 r/s when the parameter panel appears. Leave the other parameters at their default values. Click on continue. Initially both phase plot and the Poincaré plot will indicate a chaotic mode. However after a short while it will settle down to a multi-periodic mode. This is typical behavior of a system that has both chaotic and multi-periodic modes. The aim of this exercise is to get the simulated pendulum to start off in the multi-periodic mode. After the simulated pendulum has settled down to the multi-periodic mode, pick a point on its path and note the phase velocity and the phase angle. Return to the parameter panel and set the initial angular velocity and the angle to the values that you have read off the phase plot. Re-start the simulation. If all is well it will start in the multi-periodic mode. However, this is not an easy exercise and you may have to have several attempts to make it work.

6. **Three dimensional display of multiple Poincaré plots**

This exercise will show that the Poincaré plot that is displayed on the screen is a slice from a three-dimensional plot of angle, angular velocity and drive phase of the source. Click on the simulation button and choose to record a Poincaré data to a file. In order to speed things up choose computer time in the parameter panel and then choose 1000 data points in the next panel. Repeat for drive phase angles of 0.4, 0.8, 1.2, 1.6, 2.0, 2.4, 2.8 and 3.2. Plot these files to the screen in turn and print each one. In order to view the plots you might like to try gluing each to separate cards. Attach each card in order of drive phase in a vertical position to a horizontal base and separate each from each other by about an inch. View all the plots from above and at an angle of about 45 degrees and you will get an impression of the total three dimensional Poincaré plot. Now use the software to calculate the capacity dimension of each plot. You should find the values for each plot differ from each other by only a few percent.

7. **Dimension**

Although the included software can calculate capacity dimension it is instructive to carry out a 'box-counting' calculation using pencil and paper as follows. Use either the experimental pendulum or the simulation to generate a Poincaré plot with about a thousand points. Print ten copies of the data. On the first copy draw lines that divide the whole plot into four equal boxes. The length, ε, of the side of a box is 0.5. Count the number of boxes $N(\varepsilon)$ that contain at least one point. Repeat with 16 equal boxes. The length of the side of one box is 0.25. Then with 64 boxes and so on where length of the side of one box is 0.125. Continue in this manner until your patience runs out. Make a plot of $\mathrm{Ln}(N(\varepsilon))$ versus $\mathrm{Ln}(1/\varepsilon)$. This plot should have a region that is linear and the slope of this linear region is the box-counting dimension. Compare your answer to that calculated by the software.