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Advanced Physics Lab

Chaotic Pendulum

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A

Very good report! Animated Poincaré section are great!

A couple of cool applets from Rubin H Landau's site [2].

Introduction

For a system to exhibit chaotic behavior the system must have three independent dynamical variables and the equations of motion must contain some term which is non-linear or couples a number of the variables. A damped driven pendulum is one of the simplest systems that satisfies these conditions.

The equation of motion for a damped driven pendulum is

$$\text{(1)} \qquad I\theta'' + b\theta' + w\,r\,\text{Sin}[\theta] = g\,\text{Cos}[\omega_D t + \phi]$$

where **I** is the moment of inertia of the pendulum, **b** is the damping coefficient, **w** is the effective weight of the pendulum, **r** is the effective radius, **g** is the driving amplitude, ω_D is the driving frequency, and ϕ is a phase constant for the driving force.

This may be broken up into three differential equations

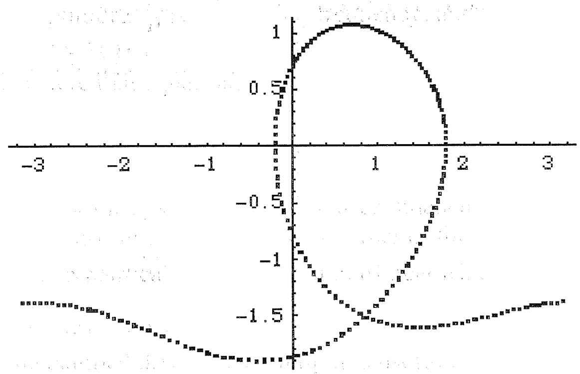
$$\frac{d\omega}{dt} = \frac{-b}{I}\omega - \mathrm{Sin}[\theta] + g\,\mathrm{Cos}[\phi]$$

$$\frac{d\theta}{dt} = \omega$$

$$\frac{d\phi}{dt} = \omega D$$

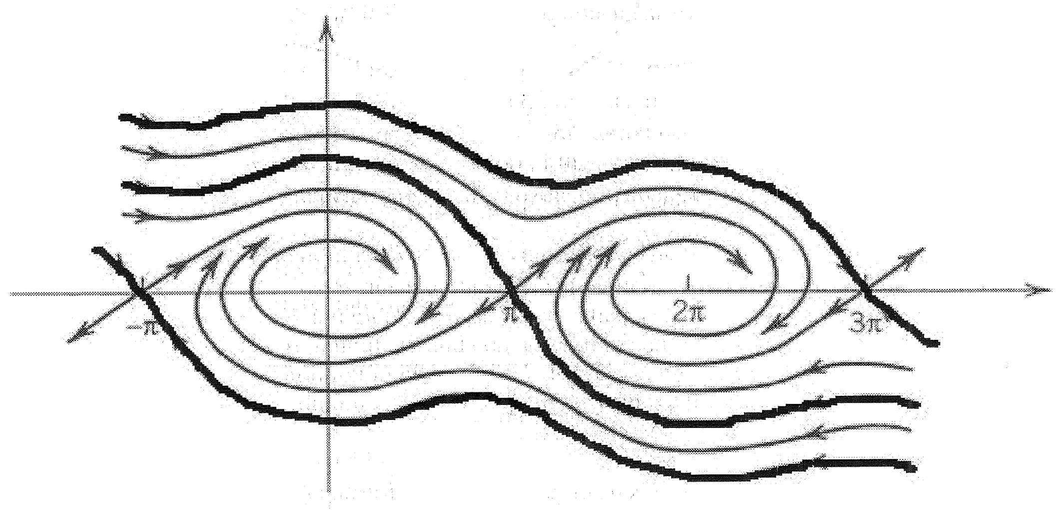
which make it clear that there are three dynamical variables with non-linear trigonometric terms.

In order to study the motions of the damped driven pendulum, it is useful to introduce the concept of phase space. In the case of the pendulum, two parameters specify the phase: the angle θ and the angular velocity ω. Hence, our phase space will be two dimensional. We will plot ω on the vertical axis and θ on the horizontal axis. A curve in phase space will be a sequence of data points parameterized by time. A typical periodic orbit in phase space looks like this:



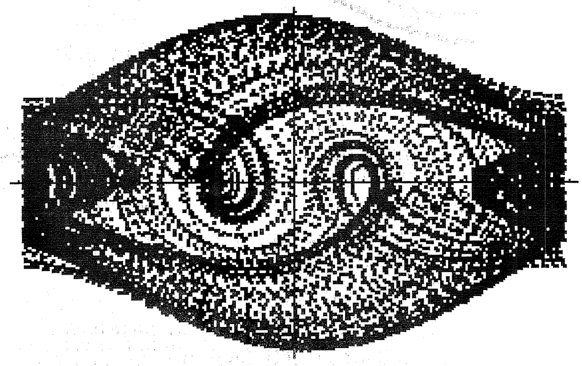
This orbit corresponds to the pendulum swinging over the top of the axis of rotation with negative ω, then moving with positive ω near the bottom of its orbit, then changing to negative ω and swinging over the top again.

Consider for a moment the damped pendulum with no driving force. The pendulum will eventually lose its kinetic energy and remain at the bottom of the orbit. This means that there are attractors at the points of the form ($2\pi k$, 0) in phase space. Each of these attractors is surrounded by a region of phase space called a *basin of attraction*. If a pendulum is started off in one of these basins of attraction, it will remain there and converge to the attractor within that basin, so the fate of the pendulum is determined by its initial condition in phase space. There are, however, highly unstable points between basins of attraction, such as the one where the pendulum is standing up directly above the axis of rotation. These unstable boundaries between basins are called *separatrices*.

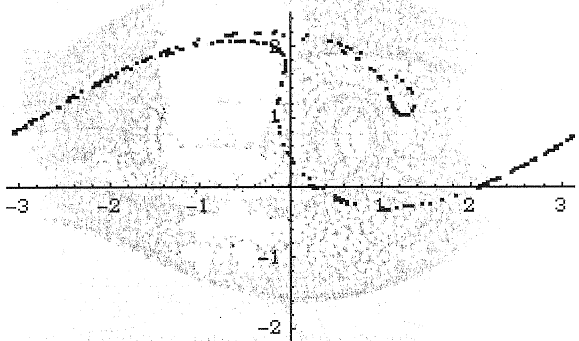


In the figure above (a modification of one from [3]), the separatrices are the bold black lines. When a drive is introduced, these separatrices become more wiggly and can become fractals which fill up phase space. When this happens, every point is a point of extreme instability, so the path of the pendulum becomes chaotic.

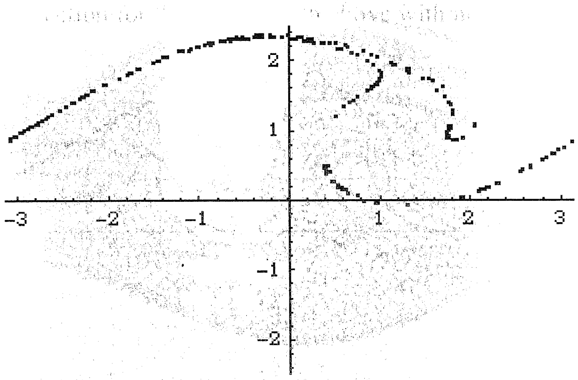
A typical path in phase space of a chaotic pendulum looks like this:



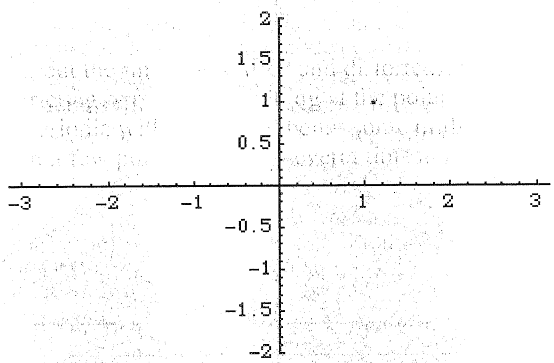
This clutters phase space quite a bit. It is useful to introduce the idea of a *poincare section* in order to analyze this behavior. Rather than looking at phase space at every instant, we look at it once every drive period. The result is a superposition of different "time-slices" of phase space. Of course, we have to specify which part of the drive period we are looking at; we have to choose a phase. The poincare section for the chaotic path above with no phase shift looks like this:



We can introduce a phase shift of $\pi/8$ to get a shifted poincare section:



The information is essentially the same, but the image is shifted and distorted. All the poincare sections look basically the same, so we don't lose anything by restricting ourselves to looking at the poincare section. Notice that if the pendulum is exhibiting periodic behavior, then it is periodic with the period being some multiple of the drive period. Thismeans that the poincare section is only going to contain a few points repeated several times. For example, the poincare section for the periodic state mentioned above is



which consists of just a single point. So the poincare section really captures everything interesting about the pendulum's motion, where periodic motion is considered not interesting.

Physical Measurements

If there is no torque signal (that is, **g** = 0), the equation of motion (1) reduces to

(2) $$I\theta'' + b\theta' + m g r \, \mathrm{Sin}[\theta] = 0$$

which, for small values of θ, has the solution

(3) $$\theta = \theta 0 \, e^{-\alpha t} \, \mathrm{Cos}[\omega 1 \, t + \phi]$$

0.011 1000

0.0141 500

where

$$\alpha = \frac{b}{2 I}, \qquad \omega 1^2 = \omega^2 - \alpha^2$$

and

$$\omega = \sqrt{\frac{m g r}{I}}.$$

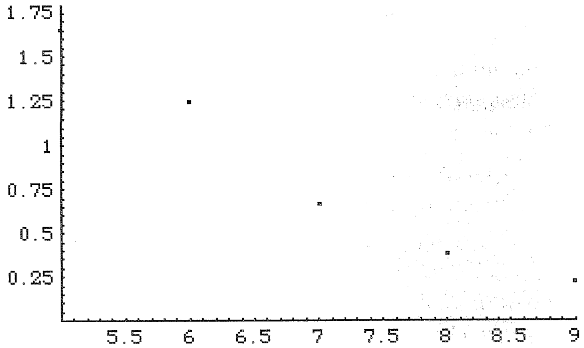
To determine the natural frequency of the pendulum, we set the damping down as far as possible and displaced the pendulum by a small angle. When we fit theresulting data, we found that the natural frequency is about 1.304 Hz, so the natural period is about 0.767 s.

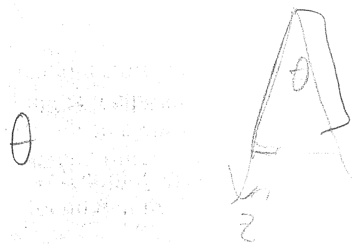
We also did a series of experiments to determine how the damping **b/I** changes as a function of the micrometer setting. For those who are compelled to look, all the fits are here. For the weak of heart, the important information is contained in the table and graph below.

micrometer setting	**b/2I** sec^{-1}
5 mm	1.6508
6 mm	1.2449
7 mm	0.6642
8 mm	0.3813
9 mm	0.2187

$I\theta''[ML^2T^{-2}] = b\theta'[ML^2T^{-1}\cdot T^{-1}]$

$\frac{b}{I} = \frac{\theta''}{\theta'} = \frac{T^{-2}}{T^{-1}} = \frac{1}{T}$



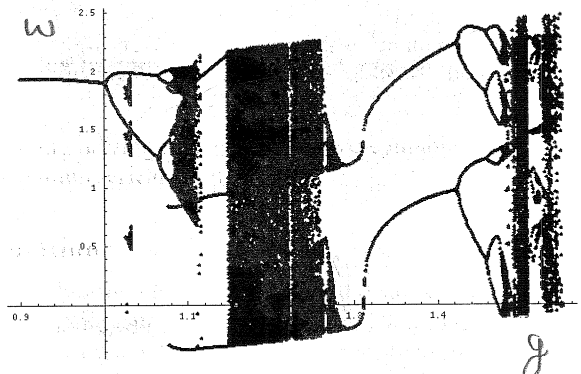


Unfortunately, due to a malfunctioning driving mechanism we were unable to collect data from the pendulum. The remaining results in this report are from numerical simulations.

Results Collected From Simulation

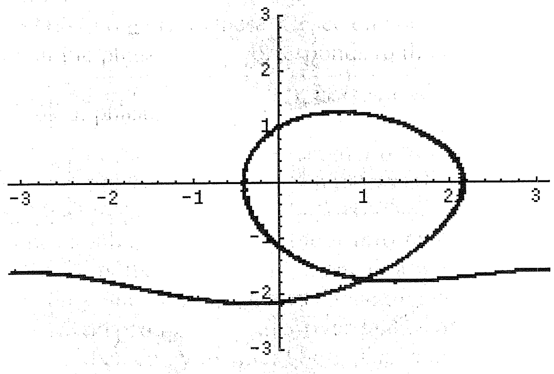
Bifurcation Diagrams

Bifurcation diagrams are a very helpful tool for determining whether or not the system is in a chaotic or periodic state. They can also tell you about the different orbits that a periodic pendulum could be in.
Here is an example of a bifurcation diagram parameters are q=2, ω=2/3: What is g?

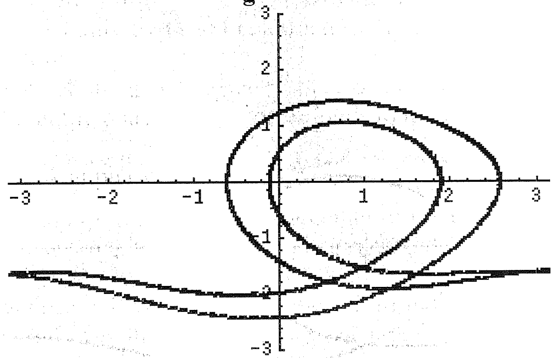


The vertical axis is angular frequency ω and the horizontal axis is a parameter for the pendulum, in the case of this diagram the horizontal axis is the driving force amplitude **g** This quickest way to explain this diagram is to say that it is a series of poincare sections where the θ-axis has been collapsed and only the ω information remains. The new horizontal axis is the gparameter for the pendulum. In the same way that we see only a one to a few points in a poincare section when the system is exhibiting periodic behavior, we have only one or a few values for angular velocity when the pendulum is periodic. Interestingly, the periodic states are in general grouped together over ranges of values for the driving force amplitude parameter, and these periodic regimes are indicated by distinct lines. The values for the parameter which give chaotic results are marked by areas thickly filled with black. In order to represent as many basins of attraction as possible, the above diagram is actually a superposition of several bifurcation diagrams with different phase parameters for the driving force. The two branches at **g**=1.4, for example, are actually two distinct attractors. The orbit ends up at one of the two attractors depending on the phase of the driving force.
We can now look at the area between **g**=1.4 and **g**=1.5, where we see one orbit breaking into two and then continuing to break into chaos. Here, we are looking at the phase which corresponds to the bottom branch of the bifurcation diagram above.
This is what we see happen when we look at phase space:

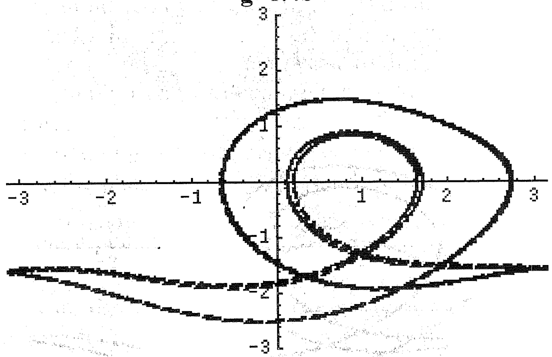
g=1.4



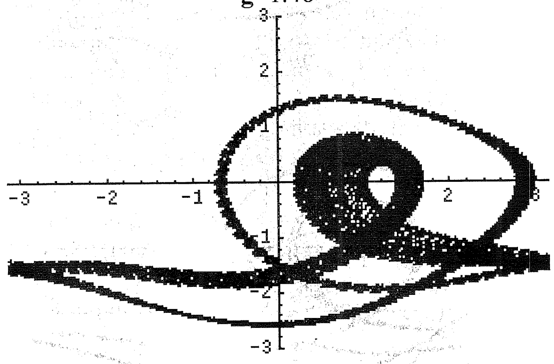
g=1.44



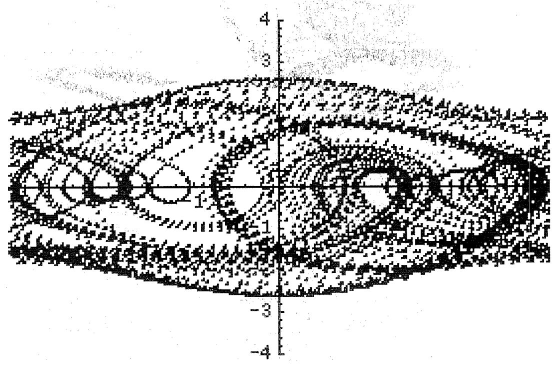
g=1.46



g=1.48

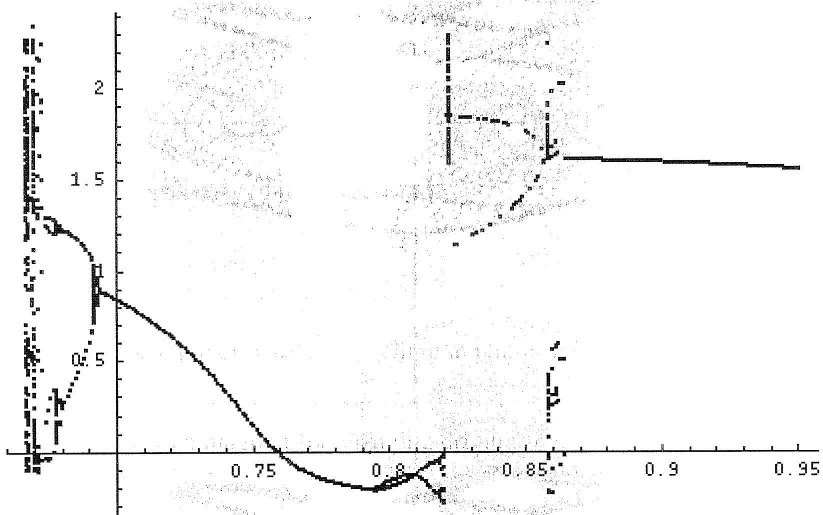


g=1.495



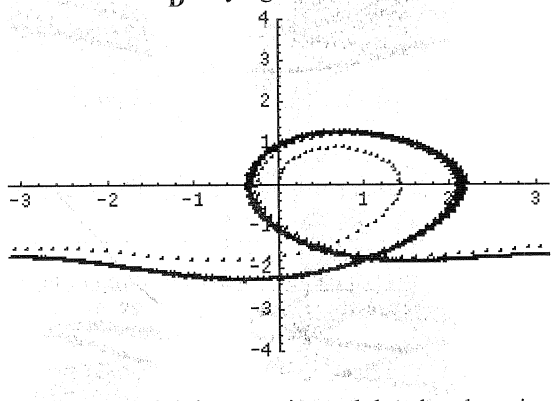
What we see here is a step by step example of what is happening in phase space as the pendulum goes from a periodic state to a chaotic one.

Another interesting case is the bifurcation diagram for changing driving frequencyω_D:



We can animate the phase diagram to watch how these single orbits split into two and into chaos:

ω_D varying from .7 to .9

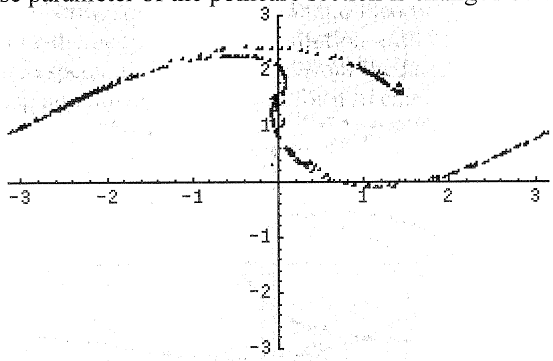


This is interesting because we can see the orbit shrink into a point and then break up into two. The sequence finishes with a circle in phase space which will continue the be the pendulum's motion until it shrinks into a point much later. It appears that these "loop" motions in phase space are especially important and from the last two examples we see the importance of the pendulum being able to flip over the top and how this is a cause for it to change orbits or move from periodic states the chaotic states.

Animated Poincare Sections

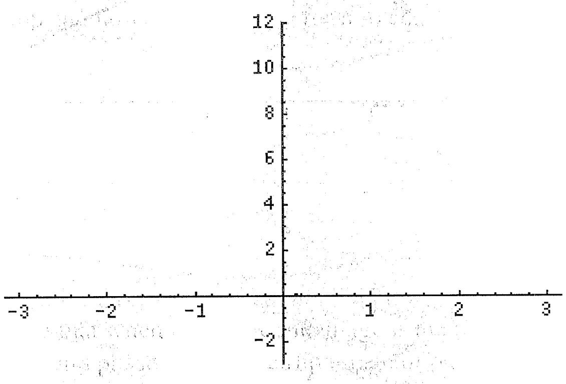
One further technique we devised which is helpful in a similar way to bifurcation diagrams is to animate the poincare sections over a changing parameter.

Here is an example of where the phase parameter of the poincare section is changed between 0 and 2π



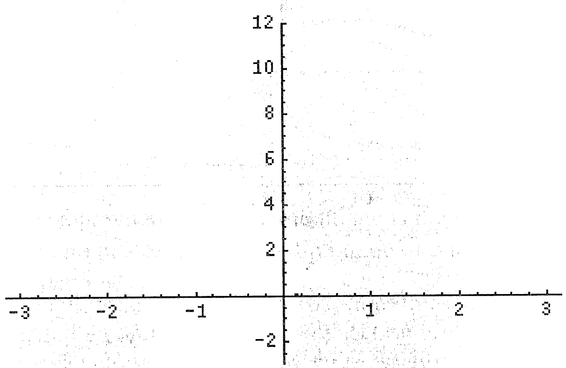
This animation is looped, but we can see that when phase = 0 the image is the the same as the image for phase = 2π. Also when the animation is half way through and phase = πit is the flip image of phase = 0.

Here is an example of animating the driving force parameter **g**from .1 to 2:



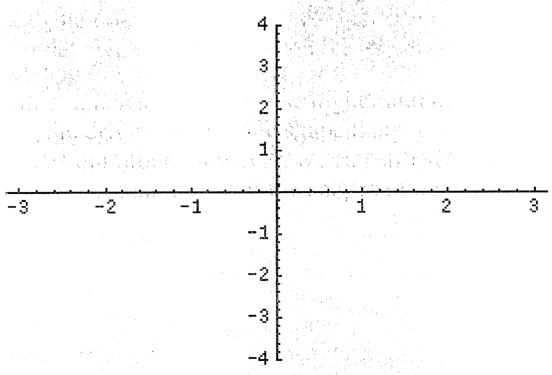
What's interesting here is that we see roughly the same shape throughout the sequence. Even when the image shows only a few points, which define a periodic state, the points in the poincare section are contained in the set of points defining the shape of a chaotic section that has yet to develop or may already have developed.

Here is an example of animating the driving force parameter **g**from .1 to 10:



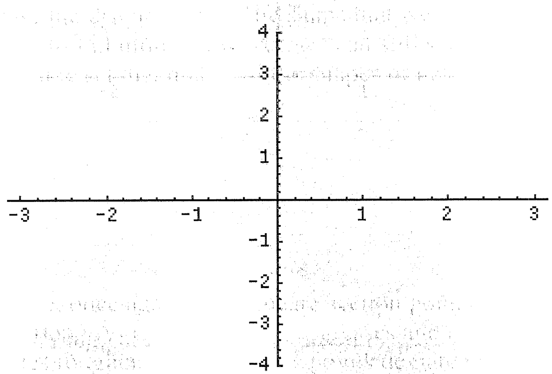
What we can see here is that the over the sequence the points rise higher and higher in phase space as we see the average momementum increases as you increase the driving force. The shape that we saw in the previous animation becomes distorted as the amplitude rises and flattens out more. However, we can still see evidence that the small number of points in periodic states appear to be part of the same set that makes of the shapes of nearby chaotic regimes.

This is an example of animating the damping parameter from .1 to 20:



What we see here is the same basic shape, once again the poincare section points appear the be part of the set of points that are part of a chaotic state's poincare section. As the animation progresses and the damping is increase what we see is an increasingly noisy shape which becomes rougher and rougher as points deviate more and more from the wavelike shape.

Here an animation showing the drive frequency parameter ω_D being changed from .0001 to 2:



As the drive frequency starts out small the pendulum appears to be in a periodic state, eventually we see a morph into the chaotic wave-like shape and after the drive frequency is about .9 it remains periodic as it increases and the motion resulting from the drive becomes smaller and smaller. From a drive frequency of about .9 and higher we see a circle in the phase diagram indicating that the pendulum is not flipping over the top and as the drive frequency increases the arc gets smaller and smaller until the pendulum appears to have no movement.

Conclusion

not really so bad!

It would have been nice to compare computed data with that collected from the real pendulum. Unfortunately we were unable to collect data from the pendulum because of a problem with the pendulum's drive. We were however, able to understand the same ideas by using the computer to calculate the motion as we would see in a real pendulum. During our exploration we learned a number of helpful ways of looking at systems and seeing them display chaos or periodicity. In examining this system we were able to identify several effects of different parameters on the system's behavior. One of the suprising results of our study was finding how well defined the shape of the poincare diagram could be in and out of chaos. Despite the chaotic nature of the system, it was still certainly possible to find some sorts of form in it. We enjoyed working on this project and found it rewarding.

References

[1] G. L. Baker and J. P. Gollub (1996), Chaotic Dynamics: an introduction2nd edition, Cambridge University Press.
[2] Rubin H Landau, http://www.physics.orst.edu/~rubin/nacphy/JAVA_pend/
[3] http://dmpeli.math.mcmaster.ca/Matlab/CLLsoftware/Pendulum/Pendulum.html