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Phase diffusion in a chaotic pendulum

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THEORY

A pendulum is taken to consist of a mass m located at a distance ℓ from a pivot and oriented at an angle θ with respect to the vertical. The total moment of inertia of the complete system, produced by m together with any other corotating components, is I. Let b be the coefficient of velocity-dependent friction. Then the equation of motion is

$$I\frac{d^2\theta}{dt^2}+b\frac{d\theta}{dt}+mg\ell\sin(\theta)=\Gamma\sin(\omega t), \tag{1}$$

where Γ and ω are the amplitude and frequency of the applied ac torque. It is conventional practice to normalize time to units of the reciprocal of the small amplitude natural frequency $\omega_0=\sqrt{mg\ell/I}$, employing overdots to signify dimensionless time derivatives, and to express torque in units of $mg\ell$. Then (1) becomes

$$\ddot{\theta}+\left[\frac{1}{Q}\right]\dot{\theta}+\sin(\theta)=\epsilon\sin(\Omega\tau), \tag{2}$$

from which it is clear that there are in reality only three independent (dimensionless) defining parameters for the driven pendulum: they are $Q=\sqrt{mg\ell I}/b$, $\epsilon=\Gamma/(mg\ell)$, and $\Omega=\omega/\omega_0$.

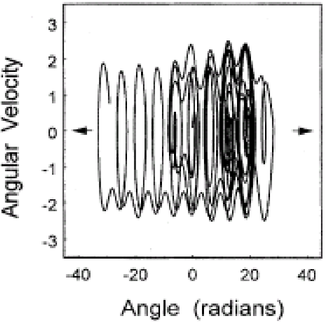

FIG. 1. Phase plane orbit for a chaotic state with $\epsilon = 0.78, \Omega = 0.62$, and $Q = 5.14$. As time proceeds, the outer limits of the orbit expand to the left and right, as suggested by the arrows. Angular velocity is measured in radians per dimensionless time unit.

As a necessary first step, a bifurcation diagram was computed for a range of dissipation coefficients between 3.0 and 7.0, these being deemed to be physically sensible values. Data were generated by sampling the evolving numerical solution of (2) once per drive cycle (Poincaré section) for a total duration of 1000 drive cycles. This was done for each of 200 Q values within the indicated range. The bifurcation diagram shown in Fig. 4 reveals a not-unexpected richness of detail, involving many periodic windows embedded within the prevalent chaos. The dissipation coefficients $Q=4.00$, 4.15, and 4.16 chosen for the previous figure can be seen to lie just inside the chaotic zone that precedes the largest periodic window.

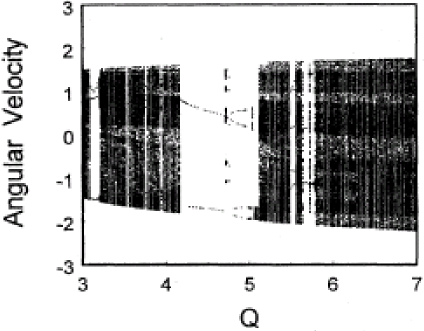

FIG. 4. A bifurcation diagram for the driven pendulum with $\epsilon = 0.78$ and $\Omega = 0.62$. Angular velocity is measured in radians per dimensionless time unit.

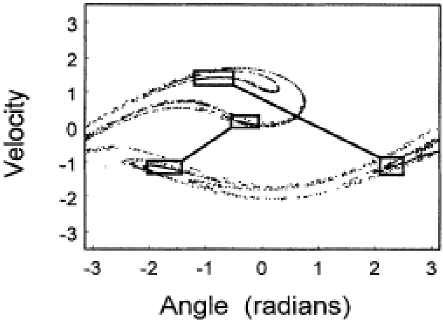

FIG. 9. Poincaré section for $Q = 5.78$. The linked boxes indicate the locations of accumulation segments for each of the period-2 orbits on the attractor. Angular velocity is measured in radians per dimensionless time unit.

Control of the chaotic driven pendulum

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A method of controlling chaos (due to Ott, Grebogi, and Yorke) is illustrated with a simulated chaotic pendulum. The method consists of stabilizing a previously unstable periodic orbit through a feedback mechanism that periodically adjusts the damping parameter of the pendulum. The presentation is pedagogical and describes the method in more detail than is typical of the research literature on controlling chaotic systems.

II. THE CHAOTIC PENDULUM

The chaotic pendulum is a driven pendulum that is operated in a parameter regime where the motion is chaotic. In dimensionless form its equation of motion may be written as

$$\frac{d^2\theta}{dt^2}+\frac{1}{q}\frac{d\theta}{dt}+\sin\theta=g\cos\omega_D t, \tag{1}$$

where θ is the angular displacement of the pendulum from the vertical, q is a friction parameter, g is the forcing amplitude, and ω_D is the forcing frequency.[3] (Small values of q imply large damping.) Variation of the parameter set (q,g,ω_D) results in various types of dynamical behavior, including chaos. The *bifurcation diagram* of Fig. 1 illustrates some of this variety. In Fig. 1 the horizontal axis shows the increase in the friction parameter q (actually a decrease in the damping) and the vertical axis shows the value of the angular velocity, $\omega=d\theta/dt$, taken at the beginning of each of many forcing cycles, *after initial transients have died away*. If the motion is periodic at the forcing frequency, then only one point occurs repeatedly for that value of q. If only a few points occur then the motion is periodic with a periodicity indicated by the number of points. For example, a period-3 window occurs at about $q=3.24$. Such periodic orbits are stable motions. If there are many points—a broad spectrum of values of ω—then the motion never repeats and is chaotic. In this case, infinitely many periodic orbits are present, but all are unstable. For this discussion we focus on the dynamics associated with a parameter set ($g=1.5$, $q=3.9$, $\omega_D=2/3$) that lies well inside the chaotic zone, as indicated in the bifurcation diagram.

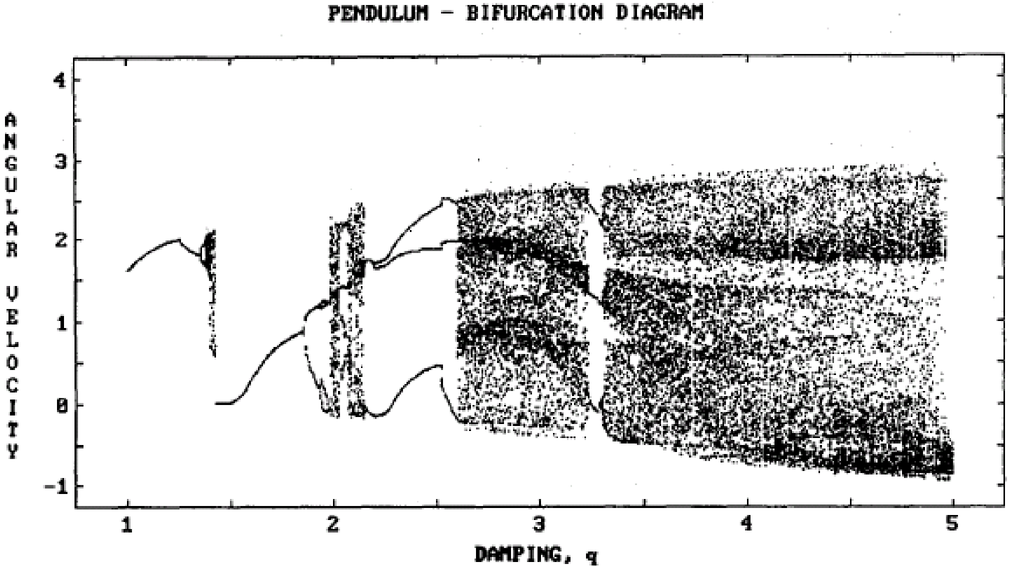

Fig. 1. A bifurcation diagram for the driven pendulum. The angular velocity ω of the pendulum at the beginning of each forcing cycle is plotted for many cycles at each value of the friction parameter q. In regions where there are many values of angular velocity, the motion is chaotic. The forcing parameter is $g=1.5$ and the forcing frequency $\omega_D=2/3$.

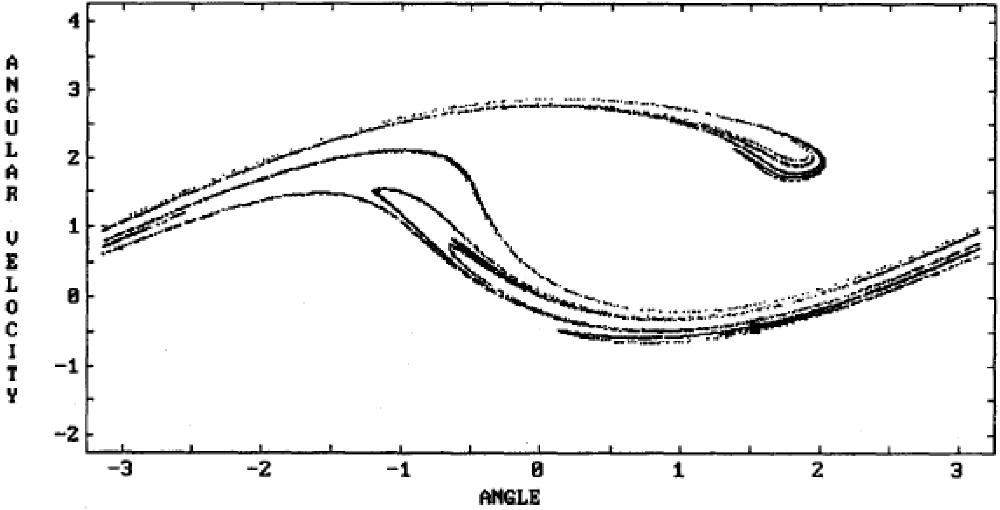

Fig. 2. A Poincaré section for the pendulum when $g=1.5$, $q=3.9$, and $\omega_D=2/3$. The values of angular velocity ω and angle θ are plotted at the beginning of each forcing cycle, for 10 000 cycles. The solid square near (1.5, −0.5) contains an unstable fixed point. (See later text for explanation.)