The Nonlinear Pendulum in Phase Space

Overview

Abstract

The nonlinear damped, driven pendulum exhibits chaotic behavior for certain values of the parameters in its equation of motion. The behavior of the nonlinear pendulum in phase space is examined for specific values of these parameters. The forcing amplitude is varied over a particular range; the trajectories undergo several period doublings and then become chaotic. Several characteristics of the system, including Poincare sections and power spectra, are examined.

Introduction and Background

The nonlinear pendulum is one example of a very simple system that can demonstrate chaotic behavior. Chaotic oscillations are of interest in many fields, including mechanical engineering, and the case of the nonlinear damped, driven oscillator may serve as a basis for mathematical models for other systems. Thus, it is interesting to explore the behavior of the nonlinear pendulum, first because it is an example of a very simple system that can demonstrate chaotic behavior, and secondly because it is mathematically similar to many other problems involving vibrations.

Chaos and the Equation of Motion

The equation of motion for the damped, driven pendulum of mass *m* and length *l* is
$ml^2\frac{\partial^2\theta}{\partial t^2} + \gamma\frac{\partial\theta}{\partial t} + mgl\sin\theta = A\cos\omega_D t$

or in dimensionless form, $\frac{\partial^2\theta}{\partial t^2} + b\frac{\partial\theta}{\partial t} + \sin\theta = a\cos\omega_D t.$

For a dynamical system described by a set of first-order differential equations, some necessary conditions for the system to exhibit chaos are (1) the system must have at least three dynamical variables, and (2) the equations of motion must have a nonlinear term that couples several of the variables. The equation of motion is written below as a system of first-order equations; we can see that it satisfies both of the above conditions.

$$\frac{d\omega}{dt} = -b\omega - \sin\theta + a\cos\phi$$
$$\frac{d\theta}{dt} = \omega$$
$$\frac{d\phi}{dt} = \omega_D \text{ (where } \phi = \omega_D t\text{)}$$

There are three variables, as well as nonlinear terms. Thus, for certain choices of the parameters, the system demonstrates chaotic behavior (Baker, 2).

Poincare Sections

How do we determine whether the motion of the system is actually chaotic for a given choice of parameters? One characteristic of a chaotic system is very sensitive dependence on initial conditions; i.e., for initial conditions very close to each other, the trajectories diverge *exponentially* with time. One way would be to determine the Lyapunov exponent, a measure of the average divergence or convergence of nearby trajectories.

Another method--the way that we use here--is the Poincare section. The phase plots reside in three-space. A Poincare section is a plane of constant phase, and thus is a two-dimensional representation of a phase plot. Because we are only looking at the two-dimensional projections of the phase plots, however, the Poincare sections appear as points.

One characteristic of a chaotic system is that although the motion is *deterministic* (given an exact initial condition, we can exactly describe its future motion mathematically) it is *nonperiodic*; i.e., it never quite repeats itself exactly. If the motion of the pendulum is periodic (nonchaotic), and we look at it once every period of the driving, we will see the pendulum take on a finite number of positions. If the motion is chaotic, and never repeats itself, we will see an infinite number of points. This is what we are doing when by taking Poincare sections; observing the behavior of the system in phase space stroboscopically and periodically. Poincare sections are also much easier to examine than the trajectories themselves, since the dimension of the Poincare mapping is one less than the dimension of the phase space.

Spectral Analysis

We can describe the motion of the pendulum as f(t), the time series of its dynamical variables. Any function f(t) can be represented as a superposition of periodic components, and we can determine the relative strength of each component; this is called *spectral analysis.* If the motion is periodic, we can express it in a Fourier series. If not, we must take a Fourier transform, using a continuum of frequencies (for a more detailed treatment of Fourier analysis, see Boas, Ch. 7). The modulus squared of the (complex-valued) Fourier transform is called the power spectrum of f(t). A discrete power spectrum is characteristic of periodic behavior, while a broad power spectrum is a good (if not perfect) indicator of chaos (Baker 59, and Szemplinska-Stupnicka, 32).

The Transition to Chaos: Period Doubling

A dynamical system, like the nonlinear pendulum, governed by a set of differential equations

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}, \ldots, m)$$

for *m* a parameter, can undergo a sequence of qualitative changes as m is varied before becoming chaotic (Szemplinska-Stupnicka, 25). There are a number of possible routes to chaos; the one we will examine is period doubling. As *m* is increased, one eigenvalue of the linearized system leaves the unit circle, passing through -1. When the eigenvalue is equal to -1, a new periodic orbit appears, with period twice that of the original orbit. So, the phase portrait then looks like a periodic oscillation with different shapes for alternate cycles. This is called period doubling. If *m* is increased further, this new periodic orbit becomes unstable, and another period doubling will appear. If a system undergoes a few period doublings, the tendency is to continue and undergo an infinite number of them (Szemplinska-Stupnicka, 24). The case that we will examine here is variation of *a*, the forcing amplitude, with the other parameters held constant. The system undergoes several period doublings as described, and then becomes chaotic.

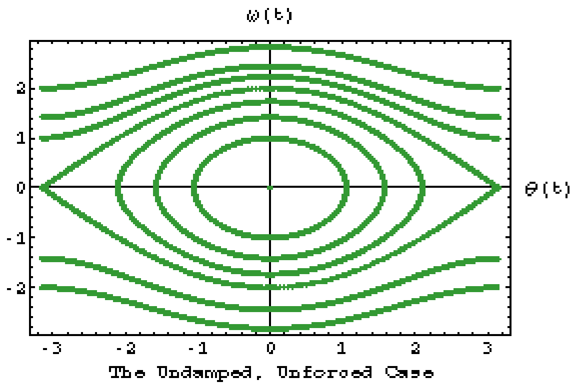
Discussion and Implementation

All plots were produced using *Mathematica*. The code is several pages long and has not been included, but is available upon request.

The Undamped, Unforced Case

We first examine a well-understood limit --the undamped, unforced pendulum. The motion in all cases has been projected onto the interval

$\theta = [-\pi, \pi]$.



This is the standard undamped, unforced case. There is a center at the origin, and saddle points at the equilibrium positions,

$$\theta = \pm\pi$$

$$\omega = 0$$

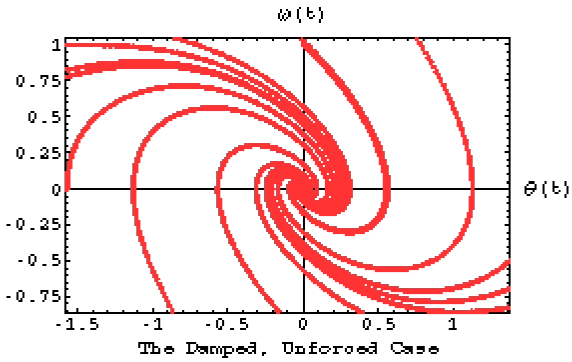
The corresponding linearized system has one positive and one negative eigenvalue at the saddle points, and a complex eigenvalue with zero real part at the center. Some of the trajectories settle down near the origins; others, with very high initial velocity, continue to orbit. This phase plot corresponds exactly to our expectations; the code seems to work.

While it appears as though the trajectories cross at the saddle points, they do not cross in three-space . . . but the 3-D phase plots are generally difficult to interpret. This is why we will examine their 2-D projections, at the cost of some information.

The undamped, unforced case is a Hamiltonian case; energy is neither dissipated by friction nor added to the system by forcing. Therefore, each of the trajectories in phase space is a curve of constant energy, i.e., a level curve of the energy function.

An Overdamped Case: b = 1

Without forcing, we increase the damping parameter to 1. Since there is no forcing, the expected trajectories should show exponentially-decaying motion towards



the origin.

As expected, we have exponential decay of the motion towards the origin. This pendulum is overdamped; it doesn't actually oscillate. The eigenvalues of the corresponding linearized system are complex (hence the oscillation) with large negative real part (causing the damping) at the equilibria.

The Transition to Chaos

Gwinn and Westervelt (1986) examined the phase plots and Poincare sections, angular velocity, and power spectra of the angular velocity for

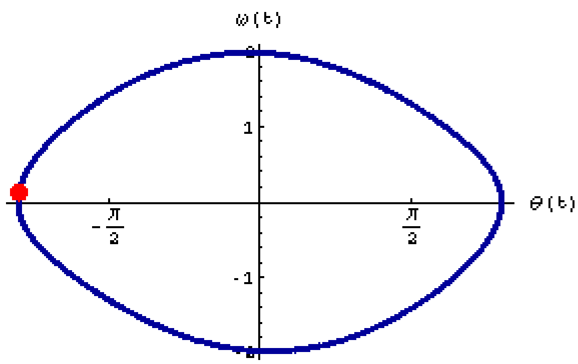
$\omega_D = \frac{2}{3}$, $b = \frac{1}{2}$, and $a = 0.9, 1.07, 1.47$, and 1.5.

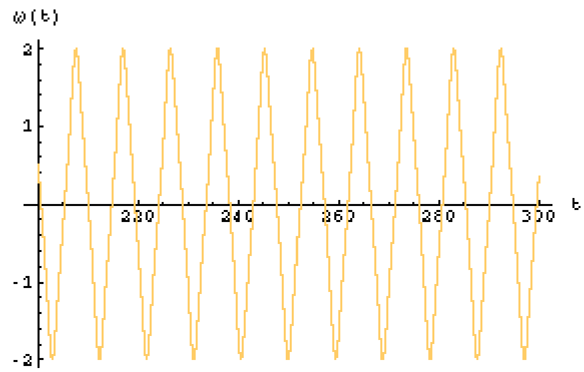
Here, we reproduce their results. The first forty or so cycles have not been shown, since we are only interested in the steady-state motion.

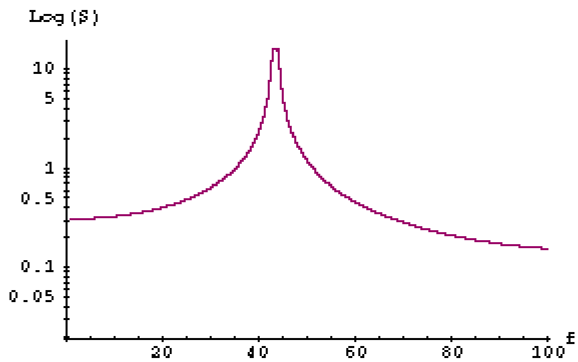
For each value of *a*, the trajectories in phase space are shown in navy blue, with the Poincare sections in red. The time series of the angular velocity are shown in yellow, with the corresponding power spectra in purple.

Plots of the FFT's are on a logarithmic scale.

a = 0.9 (Periodic)

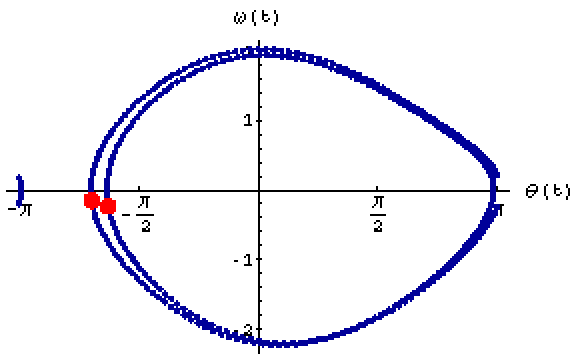


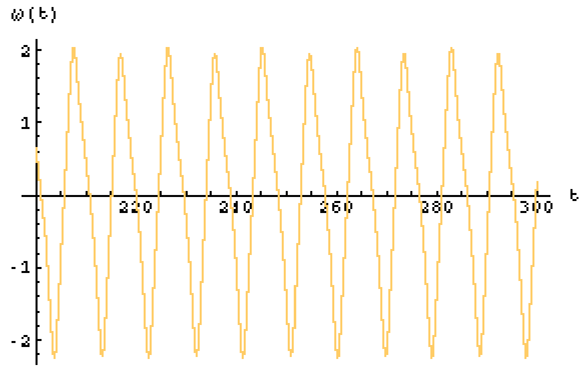


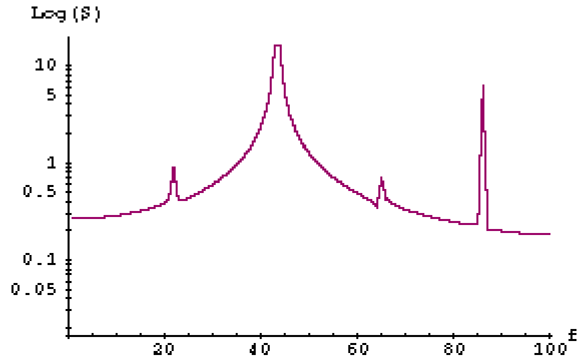


This is a periodic orbit; the pendulum is simply oscillating at the driving frequency. The Poincare section is only one point and the Fourier spectrum shows only one peak.

a = 1.07 (A Period Doubling)

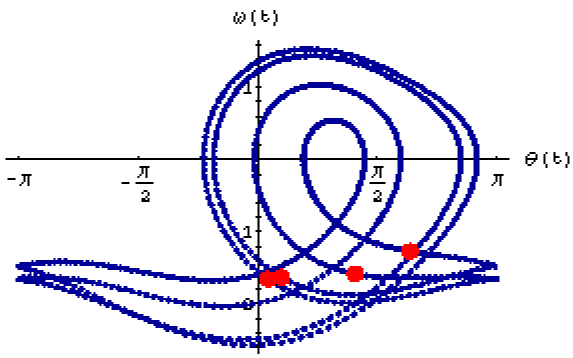


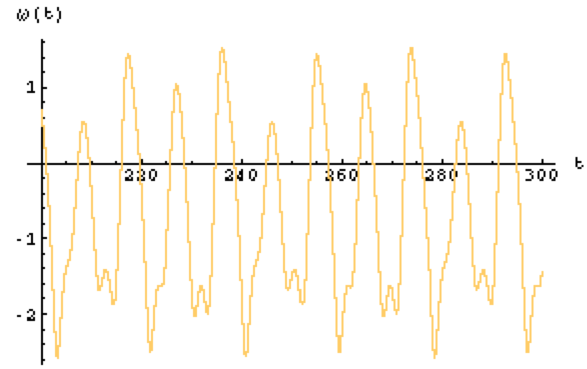


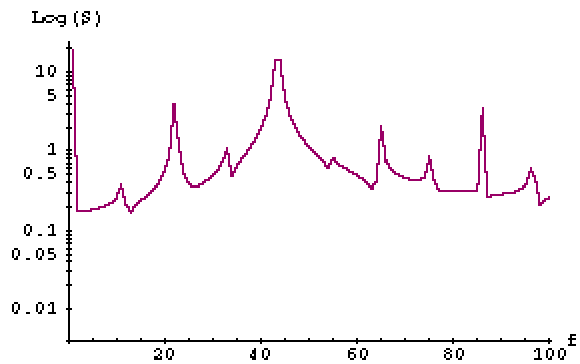


Here, we see our first period doubling. The Poincare section has increased to two points; alternate cycles of the orbit follow the same pattern. The Fourier spectrum has developed a second large peak, with two smaller subharmonics as well.

a = 1.47 (Another Period Doubling)

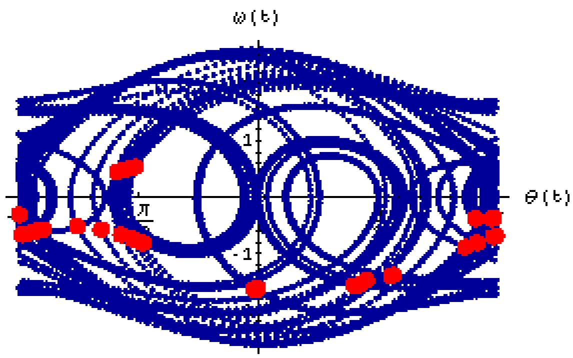


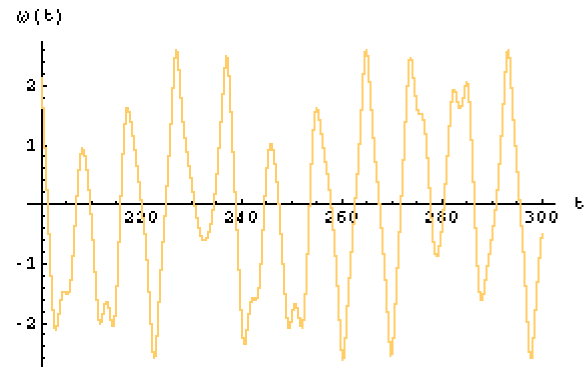


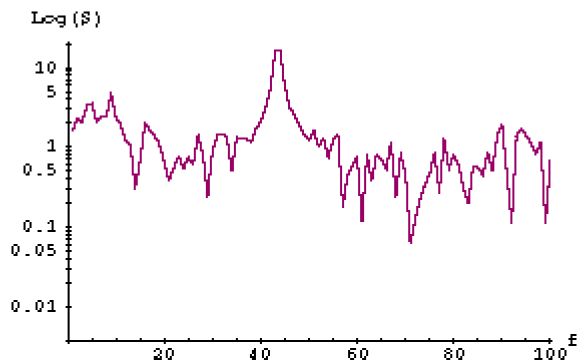


Increasing the forcing amplitude further, we observe a second bifurcation. The phase plot is now of period four; the Poincare section has grown to four points. The angular velocity becomes an increasingly complicated function of time, and its power spectrum shows an increasing number of subharmonics. Since we have found two period doublings already, it is likely that there are infinitely many, and the situation will become chaotic.

a = 1.50 (Chaotic)







This is a chaotic case. The Poincare section has an infinite number of points, the angular velocity is a complicated, nonperiodic function, and the Fourier spectrum continues to demonstrate one large peak, but is extremely noisy--a characteristic property of chaotic situations. If we continued out to infinite time, the phase plot would fill the entire region without the trajectories crossing in three-space, resulting in an attractor that folded in upon itself infinitely many times--an attractor of noninteger dimension; i.e., a fractal!

We have successfully reproduced the results of Gwinn and Westervelt as presented by Baker and Gollub, which is interesting, since the numerical methods used were entirely different. They used Runge-Kutta, while *Mathematica* uses the more precise Adams method.

Conclusions

The equation of motion for the nonlinear pendulum can be converted into a first-order system of differential equations that demonstrates some of the necessary conditions for chaotic behavior, and the system does, in fact, demonstrate chaotic behavior for specific values of the parameters.

One of the ways in which a system can become chaotic with variation of a parameter is through successive period doublings. Here, we reproduce the findings of Gwinn and Westervelt (1986) as presented by Baker and Gollub; when the driving amplitude, a, is increased from 0.9 to 1.6, the system undergoes several period doublings and eventually becomes chaotic. The Poincare sections increase from a finite to an infinite number of points, and the power spectra of the angular velocity become increasingly noisy, and finally continuous in the chaotic case.

It is interesting to note that the phase plots that Baker and Gollub obtained using the Runge-Kutta method are almost identical to those obtained here with the Adams method.

References

- Baker, G.L. and Gollub, J.P. Chaotic Dynamics: An Introduction. 2nd ed. New York: Cambridge University Press, 1996.
- Szemplinska-Stupnicka, W. and Troger, H. eds. Engineering Applications and Dynamics of Chaos. New York: Springer-Verlag, 1991.

Other Sites of Interest

- UMCP's Chaos Group; need I say more? Be sure to check out the pictures!
- A Chaotic Pendulum in Phase Space with Java; another really nifty page. This one includes an really nice animation of the pendulum.
- A .mov file of the nonlinear pendulum written by a math prof at Boston University.

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